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Embedding formulae for wave diffraction by a circular arc

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HIGHLIGHTS

- Embedding formulae of use as they reduce effort required for full characterisation of diffraction properties of scatterer.
- Formulae here for first time derived for simple curved scatterer.
- Derivation using direct approach from boundary-value problem, and also via formulation as integral equation.
- Numerical calculations demonstrate implementation and use of embedding formulae.

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ABSTRACT

For certain wave diffraction problems, embedding formulae can be derived, which represent the solution (or far-field behaviour of the solution) for all plane wave incident angles in terms of solutions of a (typically small) set of other auxiliary problems. Thus a complete characterisation of the scattering properties of an obstacle can be determined by only determining the solutions of the auxiliary problems, and then implementing the embedding formula. The class of scatterers for which embedding formulae can be derived has previously been limited to obstacles with piecewise linear boundaries; here this class is extended to include a simple curved obstacle, consisting of a thin circular arc. Approximate numerical calculations demonstrate the accuracy of the new embedding formulae.

1. Introduction

To fully characterise the wave scattering properties of an obstacle, solutions may be required for a range of plane wave incident angles. Embedding formulae are a means of reducing the effort required to achieve this full characterisation. These formulae express the solution or the far-field behaviour of the solution for an arbitrary incident wave angle in terms of analogous properties of a typically small set of other solutions. Thus once the problem is solved for this set of solutions the full characterisation follows immediately from the embedding formula without need to solve any further problems.

Embedding formulae were first derived in [1,2]. These papers showed that the solution for a plane wave incident at any angle upon a two-dimensional, thin, straight barrier containing a single gap, can be fully determined from the single solution corresponding to grazing plane wave incidence. Following [2], subsequent extensions [3–6] required the boundary-value problem to be formulated as an integral equation; the derivation of the embedding formulae then exploited the structure of the integral equation, and expressed the solution for arbitrary plane wave incident angle in terms of solutions corresponding to other plane wave incident angles. This approach was generalised in [7] in which a generalised integral equation problem, divorced from a particular wave diffraction interpretation, was addressed.

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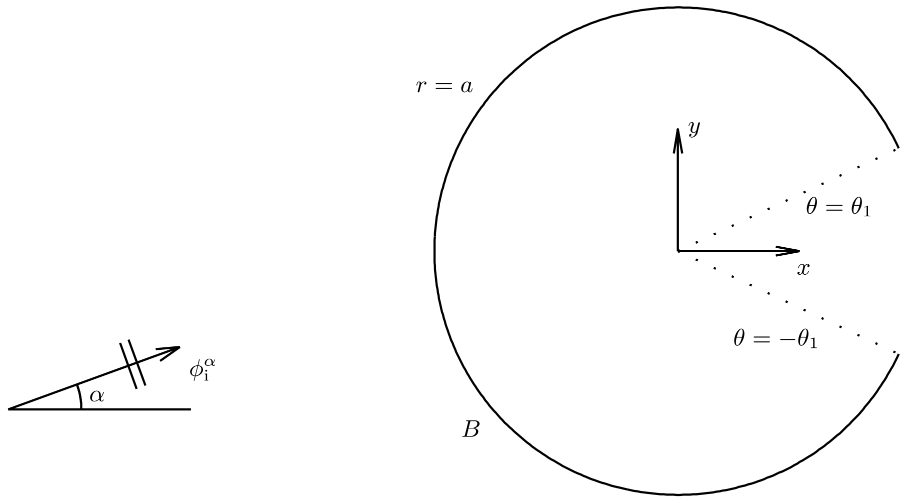

Fig. 1. Geometry of scatterer B.

The papers [8,9] instead derived embedding formulae directly from the boundary-value problem, without recourse to an integral equation formulation, and expressed the far-field of the solution for arbitrary plane wave incident angle in terms of the far-field of solutions corresponding to particular multipole forcing at the corners of the scatterers. The method was generalised to certain three-dimensional scattering problems in [10]. In many ways this approach is more versatile as the problem does not need to first be formulated as an integral equation, but the calculation of the particular solutions required for the embedding formula which are forced by source terms at the scatterer corners may not be straightforward. To address this, [11,12] modified the boundary-value problem approach to allow the far-field of the solution for arbitrary plane wave incident angle to be expressed in terms of the far-field of solutions corresponding to other plane wave incident angles.

The class of scatterers for which embedding formulae have been derived thus far is rather limited: the scatterer boundaries must be piecewise linear, with each linear portion of the boundary oriented at a rational angle (i.e. $m\pi/n$, for integers m, n) to the x-axis (say). In the current paper we extend this class of scatterers to a canonical scatterer in polar coordinates consisting of a circular arc.

Similar diffraction problems have been considered previously, though not within the context of embedding formulae. In [13] a model of a coastal harbour as a circular basin semi-embedded in an infinite coastline was developed, formulating the problem as an integral equation posed on the harbour opening and using a variational principle to provide an approximate solution. The case of porous harbour walls was considered in [14]. The problem of an electromagnetic plane wave incident upon an infinitely long, conducting, slotted cylinder is mathematically similar, and was solved numerically in [15]. In [16], the diffraction of a plane wave by precisely the scatterer geometry of the present paper was considered, though the investigation was limited to cases for which the entrance to the inner circular region was narrow, and focused on resonance excitation. More recently, [17] considered the scattering of a plane wave by a semi-circular inclusion in an otherwise infinite straight barrier.

The paper proceeds as follows. In Section 2 the boundary-value problem is introduced. In Section 3 a selection of embedding formulae is derived, firstly by adapting the approach of [11] to address the boundary-value problem directly, and then by reformulating the boundary-value problem as an integral equation, and using the results of [7] to exploit its structure. In each case the initial step is to decompose the incident plane wave $\phi_{\mathrm{i}}^{\alpha}(r,\theta) = e^{ikr\cos(\theta-\alpha)}$ into an infinite sum and then consider the problem forced by an arbitrary term in this sum (referred to below as the 'modal problem'). Approximate numerical calculations are carried out in Section 4, and a comparison is made between results determined from a direct approximation and via the embedding formulae. Finally, some conclusions and possible extensions are offered in Section 5.

2. The boundary-value problem

The scatterer takes the shape of a portion of a circular arc (see Fig. 1). Thus, in terms of standard polar coordinates (r,θ), the scatterer occupies the region

$$B = \{(r,\theta) : r = a,\ \theta \in [-\pi,\pi) \setminus (-\theta_1,\theta_1)\}, \tag{2.1}$$

in which $a > 0$ and $\theta_1 \in (0,\pi)$ are specified constants. The gap in the barrier is without loss of generality symmetrically oriented about the line $\theta = 0$. Throughout this paper we will refer to the domain for which $r < a$ as being 'within the arc', and $r > a$ as 'outside the arc'.

We suppose that there is a potential $\phi^{\alpha}(r,\theta)$ satisfying the Helmholtz equation

$$\frac{\partial^2\phi^{\alpha}}{\partial r^2} + r^{-1}\frac{\partial\phi^{\alpha}}{\partial r} + r^{-2}\frac{\partial^2\phi^{\alpha}}{\partial\theta^2} + k^2\phi^{\alpha} = 0 \quad \text{in } \mathbb{R}^2 \setminus B. \tag{2.2}$$