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Acoustic waves in fractal media: Non-integer dimensional spaces approach

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HIGHLIGHTS

- Acoustic waves in fractal media are proposed.
- General form of dimensions of region and boundary of region are suggested.
- Solution of acoustic wave equation for isotropic fractal media is obtained.
- Supersonic mode to locate fractal areas in materials is suggested.

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ABSTRACT

Acoustic waves in fractal media are considered in the framework of continuum models with non-integer dimensional spaces. Using recently suggested vector calculus for non-integer dimensional space, we consider waves in isotropic fractal media. The wave equation for non-integer dimensional space is similar to the equation of waves in non-fractal medium with power-law heterogeneity. We discuss some properties of speed of acoustic waves in fractal materials.

1. Introduction

Fractals can be considered as measurable metric sets with non-integer dimensions [1,2]. The fractal medium can be defined as a medium with a physical non-integer dimension (see [3] and references therein). The continuous model approach for description of fractal distributions of particles, charges, currents, media and fields has been proposed in [4–8] and then it has been developed in [9–12,3] and other works. This approach is based on the notion of power-law density of states [3]. We can consider fractal media as continuous media in non-integer dimensional space (NIDS). The non-integer dimension does not reflect all properties of the fractal media, but it is a main characteristic of fractal media. For this reason, continuous models with NIDS can allow us to get some important conclusions about the behavior of the fractal media.

Theory of integration in NIDS has been suggested in [13–15]. Stillinger introduces [13] a mathematical basis of integration on spaces with non-integer dimensions. A generalization of the Laplace operator for NIDS has been suggested in [13] also. Then Stillinger's approach [13] has been extended by Palmer and Stavrinou [15] to multiple variables, where a scalar Laplace operator for non-integer dimensional spaces is also suggested. These works propose differential operators of second order for scalar fields only, i.e. the scalar Laplacians in the non-integer dimensional space. The first order operators such as gradient, divergence, curl operators, and the vector Laplacian for NIDS are not considered in [13,15]. It is obvious, that consideration

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only the scalar Laplacian in NIDS-approach greatly restricts an application of continuous models with NIDS for fractal media and material. For example, we cannot use Stillinger's form of Laplacian for displacement vector field $\mathbf{u}(\mathbf{r}, t)$ in elasticity theories, and we cannot consider equations for the electric and magnetic fields of fractal media in the framework of NIDS models.

In work [16] the gradient, divergence, and curl operators are suggested only as approximations of the square of the Palmer–Stavrinou form of Laplace operator. Recently a generalization of differential vector operators of first orders (grad, div, curl), the scalar and vector Laplace operators for non-integer dimension spaces have been suggested in papers [17,18] without any approximation. This allows us to extend the application area of continuous models with NIDS. Using the suggested NIDS calculus, we can describe isotropic and anisotropic fractal media. The NIDS approach, which is based on proposed vector calculus in NIDS, has been applied in the following areas: (1) the fractal hydrodynamics to describe flow of fractal fluid in pipes [19]; (2) the fractal electrodynamics to describe fractal distribution of charges and currents [20]; (3) the theory of elasticity of fractal material [21].

In this paper, we consider acoustic waves in fractal media. Waves in fractal media have been described by continuous models in [8–12,3]. In these works the vector calculus for non-integer dimensional is not used. In this paper, we use the non-integer dimensional vector calculus, which is proposed in paper [18], to describe acoustic wave in isotropic fractal media. We prove that the wave equations for non-integer dimensional spaces are similar to the equations of waves in usual (non-fractal) media with power-law heterogeneity.

2. Vector calculus for non-integer dimensional spaces

Let us give some introduction to non-integer dimensional differentiation of integer orders (for details, see [18,17]).

In continuous models of fractal media, we will use the physically dimensionless spatial variables $x/L_0 \to x$, $y/L_0 \to x$, $z/L_0 \to x$, $\mathbf{r}/L_0 \to \mathbf{r}$, where L_0 is a characteristic size of the model. It allows us to have dimensionless differentiation in D-dimensional space such that the physical quantities of fractal media have correct physical dimensions.

For simplification, we will consider a spherically symmetric case, where scalar field Φ and vector field $\mathbf{u}$ of fractal media are independent of angles $\Phi(\mathbf{r}, t) = \Phi(r, t)$, $\mathbf{u}(\mathbf{r}, t) = u_r(r, t)\,\mathbf{e}_r$, where $\mathbf{e}_r = \mathbf{r}/r$, $r = |\mathbf{r}|$ and $u_r = u_r(r)$ is the radial component of $\mathbf{u}$. Therefore we will use the rotationally covariant functions only. This simplification is similar to simplification, which is used in [14] for integration over NIDS.

In the general case, the dimension D of region V_D of fractal media and the dimension d of boundary $S_d = \partial V_D$ of this region are not related by the equation $d = D - 1$, i.e., $\dim(\partial V_D) \neq \dim(V_D) - 1$, where $\dim(V_D) = D$ and $\dim(\partial V_D) = d$. Let us introduce new parameter $\alpha_r = D - d$, which is a dimension of fractal medium along the radial direction.

The gradient operator for the scalar field $\Phi(\mathbf{r}) = \Phi(r)$ depends on the radial dimension α_r [18] in the form

$$\mathrm{Grad}_r^{D,d}\Phi = \frac{\Gamma(\alpha_r/2)}{\pi^{\alpha_r/2}\, r^{\alpha_r-1}}\, \frac{\partial \Phi(r)}{\partial r}\, \mathbf{e}_r. \tag{1}$$

The divergence operator of the vector field $\mathbf{u} = \mathbf{u}(r)$ can be represented [18] in the form

$$\mathrm{Div}_r^{D,d}\mathbf{u} = \pi^{(1-\alpha_r)/2}\, \frac{\Gamma((d+\alpha_r)/2)}{\Gamma((d+1)/2)} \left(\frac{1}{r^{\alpha_r-1}}\, \frac{\partial u_r(r)}{\partial r} + \frac{d}{r^{\alpha_r}}\, u_r(r) \right). \tag{2}$$

These operators are (D, d)-dimensional gradient and divergence for fractal media with $d \neq D - 1$ (for details, see [18]). The curl operator of the vector field $\mathbf{u} = \mathbf{u}(r)$ is equal to zero, i.e., $\mathrm{Curl}_r^{D,d}\mathbf{u} = 0$.

Using operators (1) and (2) for the fields $\Phi = \Phi(r)$ and $\mathbf{u} = u(r)\,\mathbf{e}_r$, we get [18] the scalar and vector Laplace operators for NIDS with $d \neq D - 1$ in the form

$${}^{S}\Delta_r^{D,d}\Phi = \mathrm{Div}_r^{D,d}\mathrm{Grad}_r^{D,d}\Phi, \qquad {}^{V}\Delta_r^{D,d}\mathbf{u} = \mathrm{Grad}_r^{D,d}\mathrm{Div}_r^{D,d}\mathbf{u}. \tag{3}$$

Then the scalar Laplacian for $d \neq D - 1$ for the field $\Phi = \Phi(r)$ is defined by the equation

$${}^{S}\Delta_r^{D,d}\Phi = \frac{\Gamma((d+\alpha_r)/2)\,\Gamma(\alpha_r/2)}{\pi^{\alpha_r-1/2}\,\Gamma((d+1)/2)} \left(\frac{1}{r^{2\alpha_r-2}}\, \frac{\partial^2 \Phi}{\partial r^2} + \frac{d+1-\alpha_r}{r^{2\alpha_r-1}}\, \frac{\partial \Phi}{\partial r} \right). \tag{4}$$

The vector Laplacian in non-integer dimensional space with $d \neq D - 1$ and the field $\mathbf{u} = u_r(r)\,\mathbf{e}_r$ is

$${}^{V}\Delta_r^{D,d}\mathbf{u} = \frac{\Gamma((d+\alpha_r)/2)\,\Gamma(\alpha_r/2)}{\pi^{\alpha_r-1/2}\,\Gamma((d+1)/2)} \left(\frac{1}{r^{2\alpha_r-2}}\, \frac{\partial^2 u_r(r)}{\partial r^2} + \frac{d+1-\alpha_r}{r^{2\alpha_r-1}}\, \frac{\partial u_r(r)}{\partial r} - \frac{d\alpha_r}{r^{2\alpha_r}}\, u_r(r) \right) \mathbf{e}_r. \tag{5}$$

For $D = 3$ and $d = 2$ Eqs. (1)–(5) give the well-known expressions for the gradient, divergence, scalar Laplacian and vector Laplacian in $\mathbb{R}^3$ for fields $\Phi = \Phi(r)$ and $\mathbf{u}(r) = u_r(r)\,\mathbf{e}_r$.

The vector differential operators (1), (2), (4) and (5), which are suggested in [18] for NIDS, allow us to describe complex fractal media with the boundary dimension $d \neq D - 1$ by the NIDS approach.

The suggested operators allow us to reduce the NIDS vector differentiations to usual derivatives of integer orders with respect to $r = |\mathbf{r}|$. As a result, we can reduce partial differential equations for fields in NIDS to ordinary differential equations with respect to $r \geq 0$.