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Homogenization of some variational problems connected to the theory of lubrication

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ABSTRACT

An important problem in the theory of lubrication is to model and analyze the effects of surface roughness on the hydrodynamic performance. An efficient method to do this is homogenization. In this paper we prove a general homogenization result which allows us to consider unstationary variational problems, related to Reynolds type equations, where the lubricant may be Newtonian or non-Newtonian. Recently, the idea of finding upper and lower bounds on the effective behavior, obtained by homogenization, was applied for the first time in tribology. The homogenization result in this work may therefore also serve as a rigorous starting point for developing these successful results to unstationary problems.

1. Introduction

One important problem in tribology is to study the pressure build up in a lubricant between two surfaces (e.g. in a bearing) which are in relative motion. If the surfaces are rough, then the film thickness will oscillate rapidly both in space and time. This leads to typical homogenization problems.

Assume that the velocity of the upper surface is $V^+ = (v^+, 0)$ and of the lower surface it is $V^- = (v^-, 0)$. Moreover, the bearing domain is an open bounded subset of $\mathbb{R}^2$ denoted by Ω, the space variable $x \in \Omega$ and $t \in (0, T) \subset \mathbb{R}$ represents the time. To express the film thickness we introduce the following auxiliary function

$$h(x,t,y,\tau) = h_0(x,t) + h^+(y - \tau V_1) - h^-(y - \tau V_2),$$

where h_0, h^+ and h^- are continuously differentiable functions. Moreover, h^+ and h^- are assumed to be periodic. Without loss of generality it can also be assumed that the cell of periodicity is $Y = (0,1) \times (0,1)$ for both h^+ and h^-, i.e. the unit cube in $\mathbb{R}^2$. We also assume that v^+ and v^- are such that h is periodic in τ and we denote the cell of periodicity by Z. By using the auxiliary function h we can model the film thickness h_ε by

$$h_\varepsilon(x,t) = h(x,t,x/\varepsilon,t/\varepsilon), \quad \varepsilon > 0.$$

This means that h_0 describes the global film thickness, the periodic functions h^+ and h^- represent the roughness contribution of the two surfaces and that $\varepsilon > 0$ is a parameter which describes the roughness wavelength.

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Let the function $f = f(x, t, y, \tau, \xi)$ be periodic in y and τ. We also assume that f satisfies suitable structure conditions. Then for many types of incompressible non-Newtonian lubricants, see e.g. [7], the pressure $u_\varepsilon(x, t)$ may be obtained as the solution of a variational problem of the type (related to Reynolds type equations)

$$I_\varepsilon = \min_u \int_0^T \int_\Omega \left[f(x, t, x/\varepsilon, t/\varepsilon, \nabla_x u) + \lambda \frac{\partial h_\varepsilon}{\partial x_1} u + \gamma \frac{\partial h_\varepsilon}{\partial t} u \right] \mathrm{d}x\, \mathrm{d}t, \tag{1}$$

where λ and γ are constants. In the special case where the lubricant is Newtonian the variational problem (1) is reduced to

$$\min_u \int_0^T \int_\Omega \left[\frac{1}{2} h_\varepsilon^3 \nabla_x u \cdot \nabla_x u + \lambda \frac{\partial h_\varepsilon}{\partial x_1} u + \gamma \frac{\partial h_\varepsilon}{\partial t} u \right] \mathrm{d}x\, \mathrm{d}t. \tag{2}$$

A further simplification is obtained if one of the surfaces is smooth and unstationary while the other is rough and stationary. In that situation $h_\varepsilon = h_\varepsilon(x)$ and the governing variational problem for the pressure u_ε is of the type

$$\min_u \int_\Omega \left[\frac{1}{2} h_\varepsilon^3 \nabla_x u \cdot \nabla_x u + \lambda \frac{\partial h_\varepsilon}{\partial x_1} u \right] \mathrm{d}x. \tag{3}$$

The Lagrangians in (1) and (2) are rapidly oscillating both in space and time for small values of ε. This implies that a direct numerical analysis of these deterministic problems becomes very difficult, since a very fine mesh is needed to resolve the surface roughness. This suggests some type of averaging, which immediately leads to the concept of homogenization.

The main result of this paper is that we prove a homogenization result for a class of variational problems, which includes the homogenization of (1). Especially, this means that we show that u_ε two-scale converges to $u_0 = u_0(x, t, \tau)$, where u_0 solves a homogenized variational problem of the form

$$I = \inf_{w_0} \frac{1}{|Z|} \int_0^T \int_\Omega \int_Z \left[f_0(x, t, \tau, \nabla_x w_0) + \gamma \frac{\partial h_0}{\partial t} w_0 \right] \mathrm{d}\tau\, \mathrm{d}x\, \mathrm{d}t,$$

where

$$f_0(x, t, \tau, \xi) = \inf_w \int_Y [f(x, t, y, \tau, \xi + \nabla w) + b(x, t, y, \tau) \cdot (\xi + \nabla w)] \mathrm{d}y,$$

here b is given in terms of h, see (28), and the infimum is taken over all $w \in W^{1,p}_{\mathrm{per}}(Y)$ (functions which belong to the closure of $C^\infty_{\mathrm{per}}(Y)$ in $W^{1,p}(Y)$ and have mean value zero). Moreover, we have convergence of the energy, i.e. $I_\varepsilon \to I$ as $\varepsilon \to 0$.

The homogenization of the linear Euler equation corresponding to (2) has been studied in [3,4]. In the more engineering oriented work [3] the formal method of multiple scale expansion was used and several numerical illustrations were presented. In [4], the authors used two-scale convergence to prove the homogenization result. This is also the method, which we will develop and apply to study the homogenization of the non-linear variational problem (1). Recently, bounds on the homogenized energy density, corresponding to the linear and stationary problem (3), were derived in [9]. Later it was also clearly demonstrated in [2] that these bounds may be used to get very good approximations of the homogenized solution. It is therefore interesting to develop the idea of bounds to the more general problem (1). Hence, one additional benefit of this work is that it may serve as starting point for these further studies.

The paper is organized in the following way: In Section 2, we develop some results concerning two-scale convergence. These results are then used in Section 3 to prove a homogenization result for a class of variational problems. The result is used to homogenize (1) in Section 4. For the readers convenience we have also included an appendix concerning existence and uniqueness of minimizers to the class of variational problems, which is homogenized.

2. Two-scale convergence

The concept of two-scale convergence, see e.g. [1,11,12], is now a frequently used tool for analyzing different homogenization problems. Especially we refer the reader to[4,6,8,14], where problems related to lubrication theory have been studied. In this section, we present and prove some results about two-scale convergence, which will be used in the proof of our homogenization result.

Throughout this paper we let $x \in \Omega$ denote the space variable, where Ω is an open bounded subset of $\mathbb{R}^N$, the time variable, denoted by t, belongs to the interval $(0, T)$ and $\Omega_T = \Omega \times (0, T)$. We also assume that $1 < p < \infty$ and q is the conjugate of p, i.e. $1/p + 1/q = 1$. Without loss of generality we also assume in this section that $|Y| = |Z| = 1$.

Definition 1. Let (u_ε) be a bounded sequence in $L^p(\Omega_T)$ and $u_0 \in L^p(\Omega_T \times Y \times Z)$. Then we say that (u_ε) two-scale converges to u_0 (we write $u_\varepsilon \overset{2}{\rightharpoonup} u$) if

$$\int_0^T \int_\Omega u_\varepsilon(x, t) \phi\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) \mathrm{d}x\, \mathrm{d}t \to \int_0^T \int_\Omega \int_Y \int_Z u_0(x, t, y, \tau) \phi(x, t, y, \tau) \mathrm{d}\tau\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}t, \tag{4}$$

as $\varepsilon \to 0$, for every test function $\phi \in C_0^\infty(\Omega_T; C^\infty_{\mathrm{per}}(Y \times Z))$.