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Some exact expressions for the temporal evolution of long Rossby waves on a beta-plane

V. Nijimbere, L.J. Campbell *

School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada

HIGHLIGHTS

- We study transient forced Rossby waves in a zonal flow on a beta-plane with constant mean velocity.
- We derive two equivalent expressions for the exact time-dependent solution.
- The expressions derived are in terms of generalized hypergeometric functions.
- We investigate the late-time asymptotic behaviour of the solution.

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ABSTRACT

Some exact expressions are derived to describe the temporal evolution of forced Rossby waves in a two-dimensional beta-plane configuration where the background flow has constant zonal-mean velocity. The meridional length scale of the problem is assumed to be small relative to the zonal length scale and so the long-wave limit of zero aspect ratio is taken. In the case where the background flow velocity is zero, an exact solution is obtained in terms of generalized hypergeometric functions. A late-time asymptotic approximation is obtained and it shows that the solution oscillates with time and its amplitude goes to zero in the limit of infinite time. In the case of a non-zero background flow velocity, the solution is evaluated using two different procedures which give two equivalent expressions in terms of different generalized hypergeometric functions. The late-time asymptotic behaviour is investigated and it is found that the solution approaches a steady state in the limit of infinite time.

We also derive a solution in the form of an asymptotic series expansion for the more general situation where a Rossby wave packet is generated by a zonally-localized boundary condition comprising a continuous spectrum of wavenumbers or Fourier modes. The exact solutions found here can be used as leading-order solutions in weakly-nonlinear analyses and other studies involving more realistic configurations for time-dependent Rossby waves or wave packets.

1. Introduction

The effects of the Coriolis force give rise to large-scale oscillations in the atmosphere and ocean which are known as Rossby waves. Theoretical studies of wave dynamics and stability, in which the waves are considered to be perturbations to a basic state, have given us insight into the characteristics of Rossby wave propagation and the mechanisms by which

* Corresponding author.
E-mail address: campbell@math.carleton.ca (L.J. Campbell).

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they interact with the general circulation. Assuming that the wave amplitude is small relative to the magnitude of the large-scale flow, linear and weakly-nonlinear analytical solutions can be obtained under certain circumstances, for example, in two-dimensional barotropic configurations.

In linear barotropic models, it is often assumed that the waves are sinusoidal in time and in the zonal direction with a steady amplitude that depends only on latitude. The justification for neglecting time-dependence is that the solutions of the linear time-dependent equations generally approach a steady state in the limit of infinite time and we can therefore focus on late-time solutions and ignore the transient evolution of the waves. However, although the time-dependent terms in a linear solution may go to zero with time, their derivatives do not necessarily go to zero and could therefore make a non-negligible contribution to the solution of the nonlinear problem. In that case, a time-dependent linear solution may be needed before one can proceed to carry out a weakly-nonlinear analysis. A time-dependent linear solution can give more accurate input for further nonlinear studies and other investigations with more sophisticated models and can be used as a benchmark to test and validate the results of time-dependent numerical simulations.

However, it often proves difficult to derive an exact, rather than approximate, solution for a time-dependent configuration, even in cases where a steady solution is easily obtained. Nadon and Campbell [1] derived an exact time-dependent solution describing the evolution of forced internal gravity waves in a Boussinesq fluid with constant horizontal mean velocity and zero vertical-to-horizontal aspect ratio. The solution has proven useful for comparisons with numerical simulations [2,3]. It consists of a part with steady amplitude and a transient part expressed as a series involving Bessel functions that goes to zero in the limit of infinite time. The purpose of this paper is to derive an exact time-dependent solution for transient Rossby wave propagation on a beta-plane under the simplifying assumptions, analogous to those made in [1], that the background zonal mean velocity is constant and the meridional-to-zonal aspect ratio is zero. The waves are generated by a zonally-periodic boundary condition at the northern boundary of a rectangular domain and propagate southwards. The solutions obtained are in the form of infinite series of terms involving generalized hypergeometric functions. For these solutions to be useful for further studies, it is important to verify that the series are convergent and approach the correct limits with infinite time; we therefore investigate the late-time asymptotic behaviour of each solution and confirm that they converge to the corresponding solutions of the steady-state equation.

We then examine the case where a wave packet is generated by a zonally-localized forcing with an amplitude that varies slowly in the zonal direction. This gives a more realistic representation of the propagation of Rossby waves, particularly in the troposphere, than using a monochromatic zonally-periodic boundary condition [4]. The forcing function comprises a continuous spectrum of zonal wavenumbers with a peak at a specified central wavenumber. This form allows us to derive a solution as an asymptotic expansion from which approximate solutions can be obtained to the desired order of accuracy.

2. The time-dependent problem

The starting point is the inviscid barotropic vorticity equation which describes the evolution of a barotropic fluid flow on a beta-plane. A beta-plane is a horizontal plane tangent to the surface of the earth on which latitude is approximated by the south-to-north Cartesian coordinate y and the Coriolis parameter is approximated by a linear function of y with gradient β [5]. The zonal (west-to-east) variable is represented by x and the equation is defined on a rectangular domain on the beta-plane. The zonal extent of the domain may be taken to be a latitude circle, so that periodic boundary conditions can be applied at the east and west boundaries of the domain.

The inviscid barotropic vorticity equation is

$$\nabla^2\Psi_t + \Psi_x\nabla^2\Psi_y - \Psi_y\nabla^2\Psi_x + \beta\Psi_x = 0, \tag{1}$$

where $\Psi(x, y, t)$ is the total stream function of the fluid flow, $\nabla^2\Psi(x, y, t)$ is the total vorticity, and the subscripts denote partial differentiation with respect to time and the two space variables. All quantities appearing in the equation have been made non-dimensional using a typical velocity scale and typical length scales L_x and L_y in the zonal and meridional directions, respectively. The Laplacian operator in (1) is non-dimensional with the x-derivative in the operator multiplied by the factor $\delta = L_y^2/L_x^2$ which is the square of the aspect ratio.

A sinusoidal perturbation is generated at the northern boundary of the domain on the beta-plane and propagates southwards. The total flow is considered to be the sum of the undisturbed zonal mean flow and the perturbation. This representation is a simple model for a Rossby wave generated in the northern mid-latitude region by an external forcing mechanism such as topography or convection and propagating approximately horizontally towards the equator. Writing the perturbation stream function as $\psi(x, y, t)$ and the zonal mean velocity as $\bar{u}(y)$ and considering the amplitude of the perturbation to be small relative to that of the mean flow, we can linearize (1) and obtain

$$\left[\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right]\nabla^2\psi + (\beta - \bar{u}_{yy})\psi_x = 0. \tag{2}$$

We examine this equation on a rectangular domain given by $0 \le x \le 2\pi$, $-\infty < y < y_1$ with $t > 0$. An initial condition of zero vorticity is assumed and a zonally-periodic boundary condition

$$\psi(x, y_1, t) = e^{ikx} + \text{c.c.}, \quad k = \text{a positive integer}, \tag{3}$$