Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

# Nonlinear self-adjointness and conservation laws of the (3+1)-dimensional Burgers equation

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## HIGHLIGHTS

- Conservation laws of the (3+1)-dimensional Burgers equation were constructed.
- We establish the nonlinearly self-adjoint conditions for the Burgers equation.
- The conditions were used to obtain independent and non-trivial conserved vectors in general form.
- The conservation laws of the (3+1)-dimensional Burgers equation are infinite.

ARTICLE INFO

*Article history:*
Received 31 March 2014
Received in revised form 28 February 2015
Accepted 3 March 2015
Available online xxxx

*Keywords:*
Nonlinear self-adjointness
Conservation laws
(3+1)-dimensional Burgers' equation
Nonlinear waves

ABSTRACT

Conservation laws are one of the most important gateways to understanding physical properties of various systems. They are important for investigating integrability and for establishing existence and uniqueness of solutions. They have been used for the development of appropriate numerical methods and construction of exact solutions of partial differential equations. They play an essential role in the development of numerical methods and provide an essential starting point for finding non-locally related systems and potential variables. In the present paper we consider the (3+1)-dimensional Burgers equation whose special solitonic localized structure was investigated in Dai and Yu (2014). We show that it is nonlinearly self-adjoint and use this fact to construct infinite but independent, nontrivial and simplified conserved vectors in general form.



## 1. Introduction

Burgers' equation is the best known example of a nonlinear partial differential equation (PDE) that can be directly transformed to a linear equation. For this reason, and due to its wide range of applications, several studies have been made of generalizations of the Burgers equation in two or three spatial dimensions [1]. Among these generalizations is the (3+1)-dimensional Burgers equation

$$u_t = 2uu_y + 2vv_y + 2ww_y + u_{xx} + u_{yy} + u_{zz} \tag{1}$$

which has received considerable interest from the research community in recent years. It is commonly written in the system form

$$\begin{aligned} u_t &= 2uu_y + 2vu_x + 2wu_z + u_{xx} + u_{yy} + u_{zz}, \\ u_x &= v_y, \\ u_z &= w_y. \end{aligned} \tag{2}$$

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http://dx.doi.org/10.1016/j.wavemoti.2015.03.003
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It describes the propagation processes for nonlinear waves in fluid mechanics such as diverse non-equilibrium, nonlinear phenomena in turbulence and inter-face dynamics [2]. Researchers have used many effective methods to find various solutions of the Burgers system (2). In [2], the authors used modified mapping method to derived three families of variable separation solutions and used them to discuss interaction behaviors among four solitons in a periodic wave background. Multiple regular kink solutions and multiple singular kink solutions were derived by Abdul-Majid [3] using the Bäcklund transformation method combined with Hirota's bilinear sense. None of the analysis of Eq. (2) that has been reported so far in the literature has investigated its conservation laws, this is what we intend to address in this work.

In the study of differential equations (DEs), conservation laws play significant roles not only in obtaining in-depth understanding of physical properties of various systems, but also in the construction of their exact solutions. They describe physical conserved quantities such as mass, energy, momentum and angular momentum, as well as charge and other constants of motion. They are important for investigating integrability and linearization mappings and for establishing existence and uniqueness of solutions. They are also used in the analysis of stability and global behavior of solutions. In addition, they play an essential role in the development of numerical methods and provide an essential starting point for finding nonlocally related systems and potential variables. Moreover, the structure of conservation laws is coordinate-independent, as a point (contact) transformation maps a conservation law into a conservation law [4]. A systematic way of constructing the conservation laws of a system of DEs that admits a variational principle is via Noether's theorem [5]. Its application allows physicists to gain powerful insights into any general theory in physics just by analyzing the various transformations that would make the form of the laws involved invariant. For instance, the invariance of physical systems with respect to spatial translation, rotation, and time translation respectively give rise to the well known conservation laws of linear momentum, angular momentum and energy. But Noether's method, as elegant and powerful as it is, has several limitations. It cannot be applied to evolution equations, to differential equations of an odd order, etc. Moreover, not all local symmetries of a variational DE system are variational symmetries. To overcome these restrictions, researchers have made various generalizations of Noether's theorem [6–10]. Among those generalization, the only one associating a conservation law with every infinitesimal symmetry of an arbitrary DE is that of Ibragimov [10] which is based on self-adjointness concept. It enables one to establish the conservation laws for any DE with or without a classical Lagrangian. To widen its applicability, the concept in [10] has been extended to the notion of quasi self-adjointness [11], weak self-adjointness [12] and nonlinear self-adjointness [13,14]. We shall use the latter concept to construct non-trivial conservation laws of the Burgers system (2).

## 2. Adjoint equations

In accordance to [9,10], the formal Lagrangian of the system (2) is given by

$$\mathcal{L} = \alpha(u_t - 2uu_y - 2vu_x - 2wu_z - u_{xx} - u_{yy} - u_{zz}) + \beta(u_x - v_y) + \lambda(u_z - w_y) \tag{3}$$

where $\alpha$, $\beta$, and $\lambda$ are the new dependent variables. Writing the system (2) in the form

$$\begin{aligned} &F_1 \equiv u_t - 2uu_y - 2vu_x - 2wu_z - u_{xx} - u_{yy} - u_{zz} = 0,\\ &F_2 \equiv u_x - v_y = 0,\\ &F_3 \equiv u_z - w_y = 0, \end{aligned} \tag{4}$$

its adjoint system is defined as

$$\begin{aligned} &F_1^* \equiv \frac{\delta \mathcal{L}}{\delta u} = 0,\\ &F_2^* \equiv \frac{\delta \mathcal{L}}{\delta v} = 0,\\ &F_3^* \equiv \frac{\delta \mathcal{L}}{\delta w} = 0, \end{aligned} \tag{5}$$

where $\frac{\delta}{\delta u}, \frac{\delta}{\delta v}, \frac{\delta}{\delta w}$ are the variational derivatives with respect to $u, v, w$.

Taking into account Eqs. (3) and (5), we obtain the following adjoint equations for the Burgers system (2)

$$\begin{aligned} &\alpha_t - 2u\alpha_y - 2v\alpha_x - 2w\alpha_z + \alpha_{xx} + \alpha_{yy} + \alpha_{zz} - 2\alpha v_x + \beta_x - 2\alpha w_z + \lambda_z = 0,\\ &2\alpha u_x - \beta_y = 0,\\ &2\alpha u_z - \lambda_y = 0. \end{aligned} \tag{6}$$

These equations will be used for the derivation of nonlinearly self-adjoint condition later on.

## 3. Nonlinear self-adjointness

The concept of nonlinear self-adjointness has significantly expanded the notion of adjointness with respect to construction of conservation laws. It incorporates all the previous concepts of adjointness and thus enables more conserved vectors for DEs to be constructed. Before we proceed to establishing the general nonlinearly self-adjoint conditions for the Burgers system (2), we will like to recall the following definitions that are relevant to our work.

Please cite this article in press as: M.A. Abdulwahhab, Nonlinear self-adjointness and conservation laws of the (3+1)-dimensional Burgers equation, Wave Motion (2015), http://dx.doi.org/10.1016/j.wavemoti.2015.03.003