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On the gradient of the Green tensor in two-dimensional elastodynamic problems, and related integrals: Distributional approach and regularization, with application to nonuniformly moving sources

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HIGHLIGHTS

- The Green tensor of 2D elastodynamics and its hypersingular gradient are expressed as distributions.
- They are regularized by convolution with a source shape suitable to represent dislocation lines.
- The regularization amounts to an analytic continuation to imaginary times.
- A definite integral that gives access to fields emitted by non-uniformly moving line sources is deduced.
- The obtained closed-form expressions cover all velocity regimes, including faster-than-wave source motion.

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ABSTRACT

The two-dimensional elastodynamic Green tensor is the primary building block of solutions of linear elasticity problems dealing with nonuniformly moving rectilinear line sources, such as dislocations. Elastodynamic solutions for these problems involve derivatives of this Green tensor, which stand as hypersingular kernels. These objects, well defined as distributions, prove cumbersome to handle in practice. This paper, restricted to isotropic media, examines some of their representations in the framework of distribution theory. A particularly convenient regularization of the Green tensor is introduced, that amounts to considering line sources of finite width. Technically, it is implemented by an analytic continuation of the Green tensor to complex times. It is applied to the computation of regularized forms of certain integrals of tensor character that involve the gradient of the Green tensor. These integrals are fundamental to the computation of the elastodynamic fields in the problem of nonuniformly moving dislocations. The obtained expressions indifferently cover cases of subsonic, transonic, or supersonic motion. We observe that for faster-than-wave motion, one of the two branches of the Mach cone(s) displayed by the Cartesian components of these tensor integrals is extinguished for some particular orientations of source velocity vector.

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1. Introduction

The two-dimensional elastodynamic Green tensor [1] of the Navier equation is the primary building block of solutions for linear elasticity problems involving nonuniformly moving rectilinear line sources, such as dislocations (e.g., [2,3]). Dislocations are the fundamental carriers of plastic deformation in crystalline materials [4–6]. Mathematically, a dislocation stands as a discontinuity of the displacement field on its glide plane. This discontinuity stands as a boundary condition in traditional methods of solution of continuum mechanics [7,8], or acts more explicitly as a singular source of elastic field if the solution is tackled by means of Green functions [9]. In the latter approach, the elastodynamic solution for the strain or stress fields involves taking convolution integrals of derivatives of the Green tensor by the source functions [10,11].

Whereas the two-dimensional Green tensor itself is locally integrable, its derivatives are in general hypersingular kernels [12], namely, kernels that cannot simply be regularized by means of a Cauchy principal-value requirement. Still, they are fully legitimate objects within the theory of distributions, their apparent singularity being handled by introducing Hadamard's finite-part prescription [13–18]. From an operational standpoint, finite parts can ultimately be reduced by carrying out suitable integration by parts on convolution integrals [8]. Hypersingular kernels are commonly encountered in situations involving static [19], as well as nonuniformly moving dislocations or cracks [20,21], notably in the context of so-called boundary-element integral approaches [12,22,23], and because of their importance in practice, their handling is a recurrent issue in wave physics problems [8,23,24].

In the two-dimensional setting, the dislocation line source is transverse to the (x, y) plane of motion, in which it reduces to a point in the idealized case of a Volterra dislocation. In general, point sources lead to singular (infinite) fields at the source position and at wave fronts, which poses some problems in numerical implementations [8]. However, infinite fields are merely the hallmark of the breakdown of classical linear elasticity at the dislocation core. In reality, a physical dislocation has a finite width, that can be measured in atomistic simulations or computed by means of specially devised nonlinear models of the cohesive-zone type [25–27]. Also, dislocations of a finite width naturally arise in the framework of gradient elasticity models (e.g., [28–32]).

Being of finite width is a necessary condition for sources to undergo supersonic motion (or faster-than-light-speed motion in classical electrodynamics [24]). Indeed, faster-than-wave motions of point sources induce Mach or Cerenkov cones with unrealistically infinite field strength [24]. For dislocations or cracks, faster-than-wave motion [33] has attracted wide attention during the last decades [27,34–40]. Also, recent medical imaging techniques rely on shear-wave Mach cones induced by a fast moving ultrasonic spot at the surface of human skin [41,42]. Thus, supersonic motion must be allowed for in any comprehensive theory of radiation fields generated by moving sources. We should add that, quite generally, the concept of a point source can hardly be avoided when no information about the physical nature of the singular source of field is available. Then, Hadamard's finite part regularization, or generalizations thereof, must be employed. We refer the interested reader to Ref. [16] for a review of some recent progresses in this direction, motivated by the problem of relativistic motion of a point particle in general relativity.

However, in the specific context of dislocation theory, convoluting the point source by an appropriate shape function of finite width that represents the core provides a natural regularization of the relevant field integrals at the source location, and at the wave fronts (including Mach cones), and allows one to investigate subsonic as well as supersonic motion without the need to address these cases separately [43]. In many approaches to finite-size (so-called *smeared-out*) dislocations [44], core shape functions are often found or assumed of power-law decay in the space variable [26,45,46]. On the other hand, an exponentially-decaying shape function with cut-off characteristic length is produced by the theory of gradient elasticity of the Helmholtz type (e.g., [31,32]) where the convolution is naturally embedded within the Green function of the theory as a consequence of the constitutive relations employed.

This paper introduces an alternative power-law-type way of regularizing the elastodynamic dislocation problem, which tames all singularities of the fields in the whole (x, y) plane. While resembling certain means [46] currently employed to regularize elastostatic fields in three-dimensional simulations [46,47] it will arise, however, from an immediate analytic continuation of the Green tensor to complex values of the time variable, once the elastodynamic fundamental solutions are written down as distributions. Simplicity of implementation is indeed a necessary requirement for use in dislocation-dynamics simulations [8,46].

Section 2 reviews several different forms of the two-dimensional elastodynamic Green tensor of the Navier equation for the material displacement in an isotropic medium, and its derivatives, which we express as distributions. Their regularization is examined in Section 3, and applied in Section 4 to the computation of specific key definite integrals over time, that enter the problem of sources undergoing a velocity jump from rest to an arbitrary constant velocity, in the plane-strain and anti-plane-strain settings relevant to screw and edge dislocations, respectively. These key integrals – from which expressions of the strain and stress fields can be deduced [48] – lead, when employed for faster-than-wave source motion, to Mach cones which we further analyze here in terms of distributions. The key integrals are obtained as a difference at their time boundaries of non-trivial indefinite integrals. The latter can be used to address more general nonuniform source motions since for numerical purposes, a nonuniform motion can in general be represented conveniently as a succession of velocity jumps separating small time intervals of uniform motion [27,49]. The full solution for the fields in this problem will be reported elsewhere [48]. A discussion (Section 5) closes the paper.