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Underwater topography invisible for surface waves at given frequencies

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HIGHLIGHTS

- Linear water wave model in an infinite, two-dimensional domain is studied.
- A general method for cloaking small bottom perturbations is developed.
- The approach consists of mathematical analysis with rigorous proofs.

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ABSTRACT

We consider scattering of surface waves modeled by the linear water wave equation in an unbounded two-dimensional domain of finite depth, at a given frequency and a given incidence. Using asymptotic analysis for small perturbations of the bottom shape, we build a fixed-point equation whose unique solution is a shape which cannot be detected by a distant observer. The method works at any incidence except $\pi/4$.

1. Introduction

Related to the current progress in realizing artificial metamaterials, a great interest is devoted to different ways for achieving the cloaking of an object, making it invisible for electromagnetic waves [1]. Of course, the same question can be investigated for other types of waves, acoustic waves or water waves for instance. This has been already proved to work experimentally [2,3]. If perfect invisibility, at all frequencies and for all incident waves, remains an unreachable dream, some nice results can be obtained by considering only waves in a given frequency range. Going further, in the context of waveguides, one can take benefit of the presence at a given frequency of only a finite number of propagating waves. In other words, for a receiver located far from the perturbation, the echoes due to this later are resumed in a finite number of

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complex numbers, the so-called scattering coefficients. The question then reduces to proving the existence of perturbations canceling these coefficients. This remark has been exploited successfully in [4] for two-dimensional acoustic waveguides. However, it has been noticed that the method was rapidly deteriorating when increasing the frequency, related to the fact that the number of propagating modes is an increasing function of the frequency. This partially motivated the present study, where water waves are considered. Indeed, contrary to acoustic waveguides, there exists only one guided wave whatever the frequency is. Some other differences between the cases of acoustics and water waves will be discussed later.

More precisely, we consider a water layer of finite depth d, and we aim to find geometric distortions of the bottom which are not detectable, at a given frequency, by an observer located far from the distortion. Assuming that the perturbation of the bottom is invariant in the horizontal direction x, the scattering by an incident surface wave reduces to a two-dimensional problem set in the (y, z) cross-section of the fluid domain, with the frequency ω and the k_x component of the wave vector as parameters. We consider that the cloaking is obtained if the scattered field, due to the perturbation of the bottom, is composed of evanescent modes, so that it decreases exponentially with the distance to the perturbation. Equivalently, it means that the reflexion coefficient $\mathbf{r}$ and the transmission coefficient $\mathbf{t}$ (which are a priori complex numbers) are such that $\mathbf{r} = 0$ and $\mathbf{t} = 1$. A possible approach to find a perturbation of the bottom satisfying such conditions is to use a numerical algorithm of optimization. This has been done in [5] where the object to be cloaked is a vertical three-dimensional cylinder and in [6] where submerged steps and horizontal plates are considered in two dimensions. Our contribution here is quite different. On one hand, our results are weaker since we only get the invisibility for small perturbations of the bottom. But this is counterbalanced by the two following nice properties:

1. First, we obtain a theoretical proof of existence of invisible perturbations, which is not the case in [5] or [6]. At the same time, we prove the convergence of the fixed-point algorithm to this invisible profile.
2. Secondly, our method allows to some extent to design the main features of the perturbation, which is then only slightly modulated by the algorithm in order to achieve the perfect cloaking.

Our technique for the construction of the invisible bottom profiles h is inspired by the technique used to prove the *enforced stability* of trapped modes (or embedded eigenvalues) in [7,8]. More precisely, the method aims to build, for any given small ε, a perturbation of the bottom whose amplitude is of order ε and which is completely invisible, in the sense that the reflection $\mathbf{r}$ and the distortion of the transmission $\mathbf{t} - 1$ satisfy $\mathbf{r} = \mathbf{t} - 1 = 0$. To do that, we search the profile of the bottom perturbation h as a linear combination of a small number of given functions H_j. These functions have to fulfill some orthogonality and normalization conditions, such that the coefficients of the linear combination solve a fixed-point equation, which can be proved to be a contraction under appropriate conditions. More details will be given later, but let us mention that one can build many different invisible profiles, by changing either the value of ε (smaller than some limiting value) or by changing the functions H_j. We will show that the method works for all frequencies and all angles of incidence, except the angle $\pi/4$, where the differential of $\mathbf{r}$ with respect to h is vanishing. Explaining this exception and getting rid of this condition is an open question. A similar difficulty occurred in the case of acoustic waveguides, where it was not possible to achieve $\mathbf{t} = 1$ but only the weaker condition $|\mathbf{t}| = 1$.

A limitation of the approach is that it allows to build only small invisible perturbations of the bottom, since the contraction property is lost for large values of ε. A new idea is investigated in the present paper, to provide larger invisible perturbations. It consists of applying the previous approach, replacing the initial straight bottom by some invisible profile. Namely, one may try to apply this result repeatedly to "cultivate" an invisible profile of large amplitude. The method becomes less explicit and we are not able to prove that the degeneracy of the differential of the functional $h \longmapsto (\mathbf{r}, 1 - \mathbf{t})$ can be avoided. However, all the requirements are quite computable, and the perturbation analysis can help to develop numerical algorithms for producing invisible perturbations of larger magnitude.

The outline of the paper is the following. In Section 2, we present the method to build invisible perturbations of a straight bottom of small amplitudes. The next step where the invisible profile itself is perturbed while preserving invisibility is described in Section 3. The theoretical justifications of the asymptotic analysis used in Sections 2 and 3 are given in Section 4. Some possible extensions are finally discussed in the last section. For example we explain how to create an invisible perturbation of the bottom for a prescribed finite set of frequencies $\omega_1, \ldots, \omega_N$ or x-wave numbers $k_1, \ldots, k_J$.

2. Invisible perturbations of the bottom of small amplitude

We denote by Π the two-dimensional strip $\mathbb{R} \times (-d, 0) \ni (y, z)$, which describes the cross-section of the water domain with constant depth d. Then, for a given profile function h, we denote by Π^h the cross-section of the perturbed water domain, defined as follows (see Fig. 1):

$$\Pi^h = \{(y, z);\ y \in \mathbb{R}, 0 > z > -d - h(y)\}. \tag{1}$$

The bottom perturbation is assumed to be invariant in x, smooth and situated in the region $\{|y| < L\}$ for some $L > 0$, so that $h \in C_c^\infty(-L, L)$, which is the space of infinitely smooth functions with compact support in the segment $(-L, L)$. The main problem is formulated as finding profile functions h such that after passing over the obstacle, the surface wave of a given frequency and a given incidence produces only an exponentially decaying scattered field.

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