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Scattering by an anisotropic circle

Anders Boström

Chalmers University of Technology, Department of Applied Mechanics, SE-412 96 Göteborg, Sweden

HIGHLIGHTS

- A series solution is obtained for the wave modes inside an anisotropic circle.
- An analytical solution is given for the scattering by an anisotropic circle.
- The anisotropy has strong effects on the scattering except at low frequencies.

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ABSTRACT

The scattering by a circle is considered when the outside medium is isotropic and the inside medium is anisotropic (orthotropic). The problem is a scalar one and is phrased as a scattering problem for elastic waves with polarization out of the plane of the circle (SH wave), but the solution is with minor modifications valid also for scattering of electromagnetic waves. The equation inside the circle is first transformed to polar coordinates and it then explicitly contains the azimuthal angle through trigonometric functions. Making an expansion in a trigonometric series in the azimuthal coordinate then gives a coupled system of ordinary differential equations in the radial coordinate that is solved by power series expansions. With the solution inside the circle complete the scattering problem is solved essentially as in the classical case. Some numerical examples are given showing the influence of anisotropy, and it is noted that the effects of anisotropy are generally strong except at low frequencies where the dominating scattering only depends on the mean stiffness and not on the degree of anisotropy.

1. Introduction

The 2D scattering of a wave from a circle is an old problem in mathematical physics, see e.g. the classical book by Morse and Feshbach [1]. It requires that both the medium outside and inside the circle is homogeneous and isotropic (or cylindrically anisotropic), although a layered circle may also be considered (or a void or rigid inclusion). The scattering by anisotropic (with fixed directions of anisotropy) objects is, however, also of great interest. In mechanics this may be the scattering by fibres in a composite, the grains in a metal, or, on a larger scale, an anisotropic formation on the ground.

Waves in anisotropic media have mostly been treated in Cartesian coordinates. It is then straightforward to investigate the propagation of waves in layered media. For bounded anisotropic media (with fixed directions of anisotropy) little has been done, and most interest for such problems seems to arise for electromagnetic problems, not mechanical ones. Thus Ren [2] has derived the cylindrical and spherical wave functions in anisotropic electromagnetic media and these were used by Wu and Ren [3] to investigate the scattering by an anisotropic circle in an isotropic medium. They have been further used to treat the scattering by a sphere, see e.g. Wan and Li [4], and to derive the null field approach (T matrix method), see Doicu [5] and Wang et al. [6]. The basic idea in the derivation of these wave functions is a plane wave expansion that is

E-mail address: anders.bostrom@chalmers.se.

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transformed into polar or spherical coordinates, which leads to quite complicated expressions involving integrals that have to be computed numerically. It seems that the method has not been used for mechanical scattering problems. Very recently Zharnikov and Syresin [7] developed a very different approach (leading to a Riccati equation for the impedance operator) which they applied to the determination of the modes in an anisotropic elastic waveguide. For cylindrical orthotropy, on the other hand, more investigations have been performed; an interesting example is given by Martin and Berger [8], who investigate the eigenfrequencies in a wooden pole using somewhat similar ideas as in the present paper.

In this paper the scattering by an anisotropic circle is treated in the scalar case. This is phrased in terms of mechanical waves so it is antiplane shear waves that are assumed, i.e. the displacement is perpendicular to the plane of the circle and the problem is 2D. In contrast to the methods mentioned above the starting point is to state the anisotropic wave equation in polar coordinates. The equation then becomes more complicated than in rectangular coordinates in that the azimuthal angle φ appears in some places, but only as factors $\cos 2\varphi$ or $\sin 2\varphi$. Expanding the field in a trigonometric series in φ leads to a set of coupled ordinary differential equations. These can be solved by a power series ansatz, which leads to a very efficient way of calculating the field inside the anisotropic circle. In the isotropic medium outside the circle the classical expansions of the incident and scattered fields in terms of Bessel and Hankel functions, respectively, are made and invoking the boundary conditions this solves the problem, although it is noted that the stress boundary condition leads to a coupling between different azimuthal orders.

2. Problem formulation

Consider the scattering by an anisotropic circle of radius a residing in an isotropic infinite medium in the simplest possible setting, i.e. let the incoming field be an antisymmetric plane shear wave. Introduce a rectangular coordinate system xy with the origin at the centre of the circle and the x and y axes along the principal directions of the anisotropic medium. Also polar coordinates $r\varphi$ in the xy plane are used. The infinite medium has density ρ_0 and shear modulus μ_0. The anisotropic medium has density ρ and the shear moduli c_1 and c_2 with respect to the x and y directions, respectively. Time harmonic conditions are assumed with the time factor $\exp(-\mathrm{i}\omega t)$, where t is time and ω the angular frequency. The wave number in the infinite medium is then $k_0 = \omega\sqrt{\rho_0/\mu_0}$.

The displacement field has only an out-of-plane component u that in the medium outside the circle satisfies the 2D Helmholtz equation

$$\nabla^2 u + k_0^2 u = 0. \tag{1}$$

The medium inside the circle is assumed to be orthotropic with the plane of the circle as a symmetry plane so that the constitutive equations are

$$\sigma_{xz} = 2c_1\epsilon_{xz}, \tag{2}$$

$$\sigma_{yz} = 2c_2\epsilon_{yz}. \tag{3}$$

The shear stresses are σ_{xz} and σ_{yz} and ϵ_{xz} and ϵ_{yz} are corresponding shear strains. The equation of motion inside the circle is then

$$c_1\frac{\partial^2 u}{\partial x^2} + c_2\frac{\partial^2 u}{\partial y^2} + \rho\,\omega^2 u = 0. \tag{4}$$

The boundary conditions at $r = a$ between the isotropic medium outside the circle and the anisotropic one inside the circle are that the displacement u and the shear stress σ_{rz} are continuous. The incoming field u^{in} is taken as a plane wave with unit amplitude propagating in a direction making the angle φ_0 with the x axis

$$u^{\mathrm{in}} = \exp(\mathrm{i}k_0 r\cos(\varphi-\varphi_0)). \tag{5}$$

To fully specify the scattering problem the scattered field $u^{\mathrm{sc}} = u - u^{\mathrm{in}}$ must satisfy radiation conditions.

3. Solution inside the circle

Because the anisotropic medium resides within a circle it is convenient to formulate the equation of motion in polar coordinates. Expressed in terms of stresses this equation is

$$\frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\varphi z}}{\partial\varphi} + \frac{\sigma_{rz}}{r} + \rho\,\omega^2 u = 0. \tag{6}$$

Using the transformations for the stresses and strains between the two coordinate systems and the definition of the strains, the stresses can be given in polar coordinates as

$$\sigma_{rz} = \frac{\partial u}{\partial r}\left(c_1\cos^2\varphi + c_2\sin^2\varphi\right) + \frac{1}{r}\frac{\partial u}{\partial\varphi}\left(c_2 - c_1\right)\sin\varphi\cos\varphi, \tag{7}$$

$$\sigma_{\varphi z} = \frac{\partial u}{\partial r}\left(c_2 - c_1\right)\sin\varphi\cos\varphi + \frac{1}{r}\frac{\partial u}{\partial\varphi}\left(c_1\sin^2\varphi + c_2\cos^2\varphi\right). \tag{8}$$