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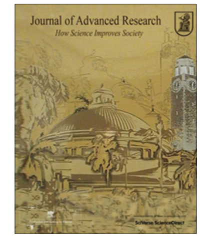

ORIGINAL ARTICLE

A study on the empirical distribution of the scaled Hankel matrix eigenvalues

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A B S T R A C T

The empirical distribution of the eigenvalues of the matrix $\mathbf{XX^T}$ divided by its trace is evaluated, where $\mathbf{X}$ is a random Hankel matrix. The distribution of eigenvalues for symmetric and nonsymmetric distributions is assessed with various criteria. This yields several important properties with broad application, particularly for noise reduction and filtering in signal processing and time series analysis.

Introduction

Consider a one-dimensional series $Y_N = (y_1, \ldots, y_N)$ of length N. Mapping this series into a sequence of lagged vectors with size L, $X_1, \ldots, X_K$, with $X_i = (y_1, \ldots, y_{i+L-1})^T \in \mathbf{R}^L$ provides the trajectory matrix $\mathbf{X} = (x_{i,j})_{i,j=1}^{L,K}$, where $L(2 \leqslant L \leqslant N/2)$ is the window length and $K = N - L + 1$;

$$\boldsymbol{X} = [X_1, \ldots, X_K] = (x_{i,j})_{i,j=1}^{L,K} = \begin{bmatrix} y_1 & y_2 & \cdots & y_K \\ y_2 & y_3 & \cdots & y_{K+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_L & y_{L+1} & \cdots & y_N \end{bmatrix}.$$

The trajectory matrix $\boldsymbol{X}$ is a Hankel matrix as has equal elements on the antidiagonals $i + j = const$. The importance of $\boldsymbol{X}$ and its corresponding singular values can be seen in different areas including time series analysis [1,2], biomedical signal processing [3,4], mathematics [5], econometrics [6] and physics [7]. However, the distribution of eigenvalues/singular values and their closed form has not been studied adequately [8]. For recent work on the generalized eigenvalues of Hankel random matrices see Naronic article [9]. For the eigenvalue distributions of beta-Wishart matrices which is a special case of random matrix see Edelman and Plamen study [10].

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Furthermore, such Hankel matrix $\boldsymbol{X}$ naturally appears in multivariate analysis and signal processing, particularly in Singular Spectrum Analysis, where each of it column represents the L-lagged vector of observations in $\mathbf{R}^L$ [11,12]. Accordingly, the aim was to determine the accurate dimension of the system, that is the smallest dimension with which the filtered series is reconstructed from a noisy signal. In this case, the main analysis is based on the study of the eigenvalues and corresponding eigenvectors. If the signal component dominates the noise component, then the eigenvalues of the random matrix $\boldsymbol{X}$ have a few large eigenvalues and many small ones, suggesting that the variations in the data takes place mainly in the eigenspace corresponding to these few large eigenvalues. Note that the number of correct singular values, r, for filtering and noise reduction, is increased with the increased L which makes the comparison among different choices (L, r) more difficult. Furthermore, despite the fact that several approaches have been proposed to identify the values of r [13], due to a lack of substantial theoretical results, none of them consider the distribution of singular values of $\boldsymbol{X}$. Here, we study the empirical distribution of singular values of $\boldsymbol{X}$ for different situations considering various criteria. Accordingly, the theoretical results on the eigenvalues of $\mathbf{XX^T}$ divided by its trace with a new view is considered in Main results. The empirical results using simulated data are presented in The empirical distribution of ζ_i. Some conclusions and recommendations for future research are drawn in Conclusion.

Main results

The singular values of $\boldsymbol{X}$ are the square root of the eigenvalues of the L by L matrix $\mathbf{XX^T}$, where $\mathbf{X^T}$ is the conjugate transpose. For a fixed value of L and a series with length N, the trace of matrix $\mathbf{XX^T}$, $tr(\mathbf{XX^T}) = \|X\|_F^2 = \sum_{i=1}^{L}\lambda_i$, where $\|\|_F$ denotes the Frobenius norm, and $\lambda_i (i = 1, \ldots, L)$ are the eigenvalues of $\mathbf{XX^T}$. Note that the increase of sample size N leads to the increase of λ_i which makes the situation more complex. To overcome this issue, we divide $\mathbf{XX^T}$ by its trace $(\mathbf{XX^T}/\sum_{i=1}^{L}\lambda_i)$, which provides the following properties.

Proposition 1. *Let $\zeta_1, \ldots, \zeta_L$ denote eigenvalues of the matrix $(\mathbf{XX^T}/\sum_{i=1}^{L}\lambda_i)$, where $\boldsymbol{X}$ is a Hankel trajectory matrix with L rows, and $\lambda_i (i = 1, \ldots, L)$ are the eigenvalues of $\boldsymbol{XX^T}$. Thus, we have the following properties*:

1. $0 \leqslant \zeta_L \leqslant \ldots \leqslant \zeta_1 \leqslant 1$,
2. $\sum_{i=1}^{L}\zeta_i = 1$,
3. $\zeta_1 \geqslant 1/L$,
4. $\zeta_L \leqslant 1/L$.

Proof. The first two properties are simply obtained from matrix algebra and thus not provided here. The outermost inequalities are attained as equalities when, for example, $y_i = 1$ for all i. To prove the third property, the first two properties are used as follows. The second part confirms $\zeta_1 + \zeta_2 + \ldots + \zeta_L = 1$. Thus, using the first property, $\zeta_1 \geqslant \zeta_i$ $(i = 2, \ldots, L)$, we obtain $\zeta_1 + \zeta_1 + \ldots + \zeta_1 = L\zeta_1 \geqslant 1 \Rightarrow \zeta_1 \geqslant 1/L$. Similarly, for the fourth property, it is straightforward to show that $\zeta_L + \zeta_L + \ldots + \zeta_L = L\zeta_L \leqslant 1 \Rightarrow \zeta_L \leqslant 1/L$, since $\zeta_L \leqslant \zeta_i (i = 1, 2, \ldots, L-1)$, and $\sum \zeta_i = 1$. Note also that if $y_L = 1$ and $y_i = 0$ for $i \neq L$ then $\zeta_1 = \ldots, \zeta_L = 1/L$. Rational number theory can also aid us to provide more informative inequalities (for more information see [14]). □

Let us now evaluate the empirical distribution of ζ_i. In doing so, a series of length N from different distributions, is generated m times. For consistency and comparability of the results, a fixed value of L, here 10, is used for all examples and case studies throughout the paper. For point estimation and comparing the mean value of eigenvalues, the average of each eigenvalue in m runs is used; $\bar{\zeta}_i$ as defined before, $i = 1, \ldots, L$, and m is the number of the simulated series. Here we consider eight different cases that can be seen in real life examples:

(a) White Noise; WN.
(b) Uniform distribution with mean zero; $U(-\alpha, \alpha)$.
(c) Uniform distribution; $U(0, \alpha)$.
(d) Exponential distribution; $Exp(\alpha)$.
(e) $\beta + Exp(\alpha)$.
(f) $\beta + t$.
(g) *Sine wave series*; $sin(\varphi)$.
(h) $\beta + sin(\varphi) + sin(\vartheta)$,

where $\alpha = 1$, $\beta = 2$, $\varphi = 2\pi t/12$, $\vartheta = 2\pi t/5$, and t is the time which is used to generate the linear trend series.

The effect of N

In this section, we consider the effect of the sample size, N on $\bar{\zeta}_i$. Fig. 1 demonstrates $\bar{\zeta}_i$ for different values of N for cases ((a)–(c)) considered in this study. In Fig. 1, $\bar{\zeta}_i$ has a decreasing pattern for different values of N. It can be seen that, for a large N, $\bar{\zeta}_i \to 1/10$ for cases (a) and (b). Thus, increasing N clearly affects the values of $\bar{\zeta}_i$ for the white noise (a) and uniform distribution (b). However, there is no obvious effect on ζ_i for other cases. For example, for case (c), $\bar{\zeta}_1$ is approximately equal to 0.8 for different values of N, and $\bar{\zeta}_{i\neq1}$ is less than $1/10$ (see Fig. 1 (right)).

Although the pattern of $\bar{\zeta}_i$ for the uniform distribution (c) is similar to exponential case (d), but for case (c), $\bar{\zeta}_1$ is greater than $\bar{\zeta}_1$ comparing to the case (d), whilst other $\bar{\zeta}_i$ are smaller. It has been observed that $\bar{\zeta}_i$ has similar patterns for cases ((c),...,(f)). The values of $\bar{\zeta}_i$ for cases (a) and (b), where Y_N generated from a symmetric distribution, are approximately the same. The results clearly indicate that increasing N does not have a significant influence on the mean of $\bar{\zeta}_i$ for all cases except (a) and (b). As a result, if Y_N is generated from WN or $U(-1, 1)$, then increasing N will affect the value of $\bar{\zeta}_i$ significantly.

The patterns of $\bar{\zeta}_i$

Let us now consider the patterns of $\bar{\zeta}_i$ for $N = 10^5$. For the white noise distribution (a) and trend series (f), $\bar{\zeta}_i$ has different pattern. It is obvious that, for the white noise series, $\bar{\zeta}_i$ converges asymptotically to 1/10, whilst for the trend series $\bar{\zeta}_1$ is approximately equal to 1, and $\bar{\zeta}_{i\neq1}$ tends to zero. Similar results were obtained for the uniform distributions, cases (b) and (c), respectively.