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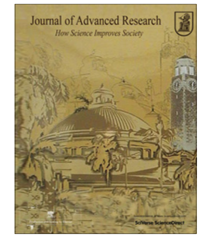

SHORT COMMUNICATION

On the boundedness and integration of non-oscillatory solutions of certain linear differential equations of second order

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In this paper, certain system of linear homogeneous differential equations of second-order is considered. By using integral inequalities, some new criteria for bounded and $L^2[0,\infty)$-solutions, upper bounds for values of improper integrals of the solutions and their derivatives are established to the considered system. The obtained results in this paper are considered as extension to the results obtained by Kroopnick (2014) [1]. An example is given to illustrate the obtained results.

Introduction

Very recently, Kroopnick [1] discussed some qualitative properties of the following scalar linear homogeneous differential equation of second order

$$x'' + a(t)x' + k^2x = 0, \quad (k \in \Re). \tag{1}$$

He established sufficient conditions under which all solutions of Eq. (1) are bounded, and the solution and its derivative are both elements in $L^2[0,\infty)$. Furthermore, the author proved that when the solutions are non-oscillatory, they approach 0 as $t \to \infty$ and calculated upper bounds for values of improper integrals of the solutions and their derivatives, that is, for $\int_0^\infty x^2(s)ds$ and $\int_0^\infty [x'(s)]^2 ds$. Finally, Kroopnick [1] introduced a short discussion about the $L^2[0,\infty)$-solutions to second order scalar linear homogeneous differential equation $x'' + q(t)x = 0$.

The results obtained by Kroopnick are summarized in Theorems A and B.

Theorem A (*Kroopnick [1, Theorem 1]*). Given Eq. (1). Suppose $a(.)$ is a positive element in $C[0,\infty)$ such that $A_0 > a(t) > a_0 > 0$ for some positive constants A_0 and a_0,

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then all solutions to Eq. (1) are bounded. Moreover, if any solution $x(.)$ is non-oscillatory, then both $x(t) \to 0$ and $x'(t) \to 0$ as $t \to \infty$. Finally, the solution and its derivative are both elements of $L^2[0, \infty)$.

The second result proved by Kroopnick [1] is the following theorem.

Theorem B (*Kroopnick [1, Theorem II]*). Under the conditions of Theorem I, the following inequalities hold:

$$\int_0^\infty [x'(s)]^2 ds \leqslant \frac{[x'(0)]^2 + k^2[x(0)]^2}{2a_0}$$

and

$$\int_0^t [x'(s)]^2 ds \leqslant x(0)x'(0) + \frac{1}{2k^2} a(0)[x(0)]^2 + \frac{[x'(0)]^2 + k^2[x(0)]^2}{2a_0 k^2}.$$

It should be noted that Kroopnick [1] proved both of Theorems A and B by the integral inequalities.

In this paper, in lieu of Eq. (1), we consider the more general vector linear homogenous differential equation of the second order of the form

$$X'' + a(t)X' + b(t)X = 0, \tag{2}$$

where $X \in \Re^n$, $t \in \Re^+$, $\Re^+ = [0, \infty)$; $a(.), b(.) : \Re^+ \to (0, \infty)$ are continuous functions and $a(.)$ and $b(.)$ have also lower and upper positive bounds.

It should be noted that Eq. (2) represents the vector version for the system of real second order linear non-homogeneous differential equations of the form

$$x'' + a(t)x_i' + b(t)x_i = 0, \quad (i = 1, 2, \ldots, n).$$

Then, it is apparent that Eq. (1) is a special case of Eq. (2).

It is worth mentioning that, in the last century, stability, instability, boundedness, oscillation, etc., theory of differential equations has developed quickly and played an important role in qualitative theory and applications of differential equations. The qualitative behaviors of solutions of differential equations of second order, stability, instability, boundedness, oscillation, etc., play an important role in many real world phenomena related to the sciences and engineering technique fields. See, in particular, the books of Ahmad and Rama Mohana Rao [2], Bellman and Cooke [3], Chicone [4], Hsu [5], Kolmanovskii and Myshkis [6], Sanchez [7], Smith [8], Tennenbaum and Pollard [9] and Wu et al. [10]. In the case $n = 1$, $a(t) = 0$ and $b(t) \neq 0$, Eq. (2) is known as Hill equation in the literature. Hill equation is significant in investigation of stability and instability of geodesic on Riemannian manifolds where Jacobi fields can be expressed in form of Hill equation system [11]. The mentioned properties have been used by some physicists to study dynamics in Hamiltonian systems [12]. Eq. (1) is also encountered as a mathematical model in electromechanical system of physics and engineering [2]. By this, we would like to mean that it is worth to work on the qualitative properties of solutions of Eq. (2).

In this paper, stemmed from the ideas in Kroopnick [1,13], Tunc [14,15] and Tunc and Tunc [16], etc., we obtain here some new criteria related to the bounded and $L^2[0, \infty)$-solutions, upper bounds for values of improper integrals of solutions of Eq. (2) and their derivatives, where the functions $a(.)$ and $b(.)$ do not need to be differentiable at any point and the Gronwall inequality is avoided which are the usual cases. The technique of proofs involves the integral test and an example is included to illustrate the obtained results. This work has a new contribution to the topic in the literature. This case shows the novelty of this work. The results to be established here may be useful for researchers working on the qualitative theory of solutions of differential equations.

The main results

In this section, we introduce the main results. We arrive at the following theorem:

Theorem 1. *Given* Eq. (2). *Suppose $a(.)$ and $b(.)$ are positive elements in $C[0, \infty)$ such that*

$$A_0 > a(t) > a_0 > 0 \quad \text{and} \quad B_0 > b(t) > b_0 > 0$$

for some positive constants A_0, a_0, B_0 and b_0 and for all $t \in \Re^+$.

Then all solutions of Eq. (2) *are bounded. Moreover, if any solution $X(.)$ of* Eq. (2) *is non-oscillatory, then both $\|X(t)\| \to 0$ and $\|X'(t)\| \to 0$ as $t \to \infty$. Finally, the solution and its derivative are both elements of $L^2[0, \infty)$.*

Proof. First, we prove boundedness of solutions of Eq. (2). When we multiply Eq. (2) by $2X'(t)$, it follows that

$$2\langle X'(t), X''(t)\rangle + 2\langle a(t)X'(t), X'(t)\rangle + 2\langle b(t)X(t), X'(t)\rangle = 0. \tag{3}$$

Integrating estimate (3) from 0 to t and then applying integration by parts to the first term on the left hand side of (3), we find

$$2\int_0^t \langle X'(s), X''(s)\rangle ds + 2\int_0^t \langle a(s)X'(s), X'(s)\rangle ds + 2\int_0^t \langle b(s)X(s), X'(s)\rangle ds = 0,$$

and

$$\|X'(t)\|^2 - \|X'(0)\|^2 + 2\int_0^t a(s)\|X'(s)\|^2 ds + 2\int_0^t \langle b(s)X(s), X'(s)\rangle ds = 0, \tag{4}$$

respectively.

In view of the last two terms included in estimate (4), first apply the mean value theorem for integrals and then use the assumptions of Theorem 1, it follows that

$$2\int_0^t a(s)\|X'(s)\|^2 ds = 2a(t^*)\int_0^t \|X'(s)\|^2 ds \geqslant 2a_0 \int_0^t \|X'(s)\|^2 ds,$$

$$\begin{aligned} 2\int_0^t \langle b(s)X(s), X'(s)\rangle ds &= 2b(t^*)\int_0^t \langle X(s), X'(s)\rangle ds \\ &= 2b(t^*)\int_0^t \langle X(s), X'(s)\rangle ds \\ &\geqslant b_0\|X(t)\|^2 - b_0\|X(0)\|^2, \end{aligned}$$