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Numerical simulation of fractional Cable equation of spiny neuronal dendrites

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ABSTRACT

In this article, numerical study for the fractional Cable equation which is fundamental equations for modeling neuronal dynamics is introduced by using weighted average of finite difference methods. The stability analysis of the proposed methods is given by a recently proposed procedure similar to the standard John von Neumann stability analysis. A simple and an accurate stability criterion valid for different discretization schemes of the fractional derivative and arbitrary weight factor is introduced and checked numerically. Numerical results, figures, and comparisons have been presented to confirm the theoretical results and efficiency of the proposed method.

Introduction

The Cable equation is one of the most fundamental equations for modeling neuronal dynamics. Due to its significant deviation from the dynamics of Brownian motion, the anomalous diffusion in biological systems cannot be adequately described by the traditional Nernst–Planck equation or its simplification, the Cable equation. Very recently, a modified Cable equation was introduced for modeling the anomalous diffusion in spiny neuronal dendrites [1]. The resulting governing equation, the so-called fractional Cable equation, which is similar to the traditional Cable equation except that the order of derivative with respect to the space and/or time is fractional.

Also, the proposed fractional Cable equation model is better than the standard integer Cable equation, since the fractional derivative can describe the history of the state in all intervals, for more details see [1,2] and the references cited therein.

The main aim of this work is to solve such this equation numerically by an efficient numerical method, fractional weighted average finite difference method (FWA–FDM).

In recent years, considerable interest in fractional calculus has been stimulated by the applications that this calculus finds in numerical analysis and different areas of physics and engineering, possibly including fractal phenomena. The applications range from control theory to transport problems in fractal structures, from relaxation phenomena in disordered

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Table 1 The absolute error of the numerical solution of Eq. (35).

x	The absolute error
0.1	0.3063×10^{-3}
0.2	0.5826×10^{-3}
0.3	0.8019×10^{-3}
0.4	0.9427×10^{-3}
0.5	0.9912×10^{-3}
0.6	0.9427×10^{-3}
0.7	0.8019×10^{-3}
0.8	0.5826×10^{-3}
0.9	0.3063×10^{-3}

media to anomalous reaction kinetics of subdiffusive reagents [2,3]. Fractional differential equations (FDEs) have been of considerable interest in the literatures, see for example [4–13] and the references cited therein, the topic has received a great deal of attention especially in the fields of viscoelastic materials [14], control theory [15], advection and dispersion of solutes in natural porous or fractured media [16], anomalous diffusion, signal processing and image denoising/filtering [17].

In this section, the definitions of the Riemann–Liouville and the Grünwald–Letnikov fractional derivatives are given as follows:

Definition 1. The Riemann–Liouville derivative of order α of the function $y(x)$ is defined by

$$D_x^{\alpha} y(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{y(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, \quad x > 0, \tag{1}$$

where n is the smallest integer exceeding α and $\Gamma(.)$ is the Gamma function. If $\alpha = n \in \mathbb{N}$, then (1) coincides with the classical n^{th} derivative $y^{(n)}(x)$.

Definition 2. The Grünwald–Letnikov definition for the fractional derivatives of order $\alpha > 0$ of the function $y(x)$ is defined by

$$D^{\alpha} y(x) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\left[\frac{x}{h}\right]} w_k^{(\alpha)} y(x - hk), \quad x \geqslant 0, \tag{2}$$

where $\left[\frac{x}{h}\right]$ means the integer part of $\frac{x}{h}$ and $w_k^{(\alpha)}$ are the normalized Grünwald weights which are defined by $w_k^{(\alpha)} = (-1)^k \binom{\alpha}{k}$.

The Grünwald–Letnikov definition is simply a generalization of the ordinary discretization formula for integer order derivatives. The Riemann–Liouville and the Grünwald–Letnikov approaches coincide under relatively weak conditions; if $y(x)$ is continuous and $y'(x)$ is integrable in the interval $[0, x]$, then for every order $0 < \alpha < 1$ both the Riemann–Liouville and the Grünwald–Letnikov derivatives exist and coincide for any value inside the interval $[0, x]$. This fact of fractional calculus ensures the consistency of both definitions for most physical applications, where the functions are expected to be sufficiently smooth [15,18].

The plan of the paper is as follows: In the second section, some fractional formulae and some discrete versions of the fractional derivative are given. Also, the FWA–FDM is developed. In the third section, we study the stability and the accuracy of the presented method. In section ''Numerical results'' numerical solutions and exact analytical solutions of a typical fractional Cable problem are compared. The paper ends with some conclusions in section ''Conclusion and remarks.''

We consider the initial-boundary value problem of the fractional Cable equation which is usually written in the following way

$$u_t(x,t) = D_t^{1-\beta} u_{xx}(x,t) - \mu D_t^{1-\alpha} u(x,t), \quad a < x < b, \quad 0 < t \leqslant T, \tag{3}$$

where $0 < \alpha, \beta \leqslant 1$, μ is a constant and $D_t^{1-\gamma}$ is the fractional derivative defined by the Riemann–Liouville operator of order $1-\gamma$, where $\gamma = \alpha, \beta$. Under the zero boundary conditions

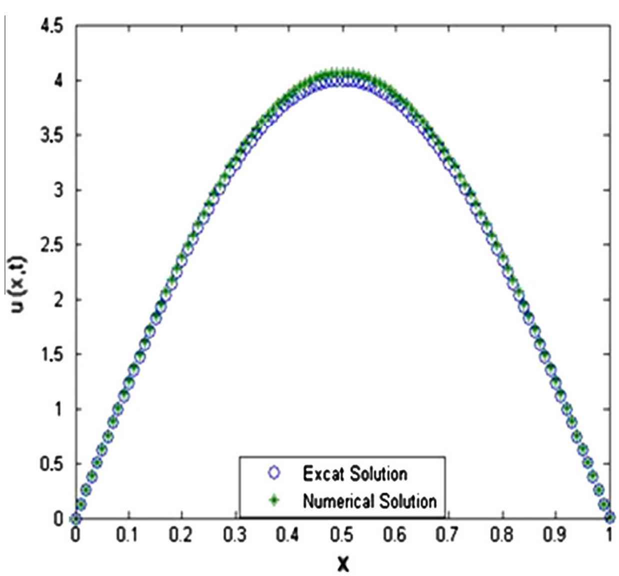

Fig. 1 The behavior of the exact solution and the numerical solution of (35) at $\lambda = 0$ for $\alpha = 0.2, \beta = 0.7, \Delta x = \frac{1}{100}, \Delta t = \frac{1}{40}$, with $T = 2$.

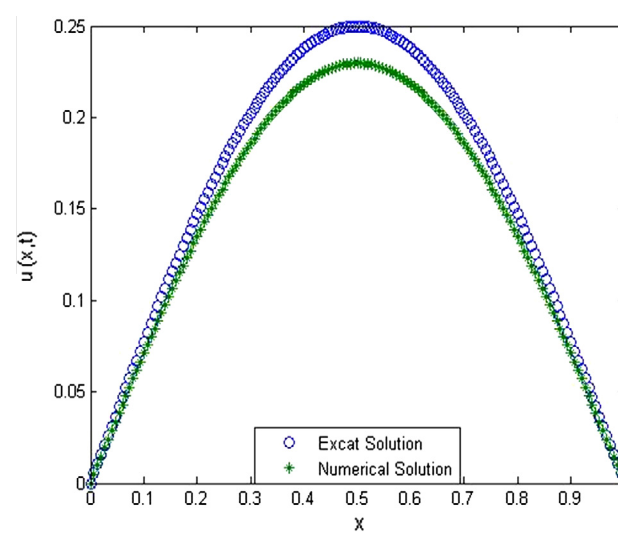

Fig. 2 The behavior of the exact solution and the numerical solution of (35) at $\lambda = 0.5$ for $\alpha = 0.1, \beta = 0.3, \Delta x = \frac{1}{150}, \Delta t = \frac{1}{10}$, with $T = 0.5$.