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ORIGINAL ARTICLE

Solving Abel integral equations of first kind via fractional calculus

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Abstract We give a new method for numerically solving Abel integral equations of first kind. An estimation for the error is obtained. The method is based on approximations of fractional integrals and Caputo derivatives. Using trapezoidal rule and Computer Algebra System Maple, the exact and approximation values of three Abel integral equations are found, illustrating the effectiveness of the proposed approach.

1. Introduction

Consider the following generalized Abel integral equation of first kind:

$$f(x) = \int_0^x \frac{k(x,s)g(s)}{(x-s)^\alpha} ds, \quad 0 < \alpha < 1, \quad 0 \leqslant x \leqslant b, \tag{1.1}$$

where g is the unknown function to be found, f is a well behaved function, and k is the kernel. This equation is one of the most famous equations that frequently appears in many physical and engineering problems, like semi-conductors, heat conduction, metallurgy and chemical reactions (Gorenflo, 1996; Gorenflo and Vessella, 1991). In experimental physics, Abel's integral equation of first kind (1.1) finds applications in plasma diagnostics, physical electronics, nuclear physics, optics and astrophysics (Knill et al., 1993; Kosarev, 1980). To determine the radial distribution of the radiation intensity of a cylinder discharge in plasma physics, for example, one needs to solve an integral Eq. (1.1) with $\alpha = \frac{1}{2}$. Another example of application appears when one describes velocity laws of stellar winds (Knill et al., 1993). If $k(x,s) = \frac{1}{\Gamma(1-\alpha)}$, then (1.1) is a fractional integral equation of order $1-\alpha$ (Podlubny, 1999). This problem is a generalization of the tautochrone problem of the calculus of variations, and is related with the born of fractional mechanics (Riewe, 1997). The literature on integrals and

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derivatives of fractional order is now vast and evolving (see, e.g., Diethelm et al., 2005; Diethelm and Freed, 2002; Tarasov, 2013; Tenreiro Machado et al., 2011; Wang et al., 2011). The reader interested in the early literature, showing that Abel's integral equations may be solved with fractional calculus, is referred to Gel'fand and Shilov (1964). For a concise and recent discussion on the solutions of Abel's integral equations using fractional calculus see Li and Zhao (2013).

Many numerical methods for solving (1.1) have been developed over the past few years, such as product integration methods (Baker, 1977; Baratella and Orsi, 2004), collocation methods (Brunner, 2004), fractional multi step methods (Lubich, 1985, 1986; Plato, 2005), backward Euler methods (Baker, 1977), and methods based on wavelets (Lepik, 2009; Saeedi et al., 2011a,b). Some semi analytic methods, like the Adomian decomposition method, are also available, which produce a series solution (Bougoffa et al., 2013). Unfortunately, the Abel integral Eq. (1.1) is an ill-posed problem. For $k(x,s)=\frac{1}{\Gamma(1-\alpha)}$, Gorenflo (1996) presented some numerical methods based on fractional calculus, e.g., using the Grunwald–Letnikov difference approximation

$$D^{\alpha}f\simeq h^{-\alpha}\sum_{r=0}^{n}(-1)^{r}\binom{\alpha}{r}f(x-rh). \tag{1.2}$$

If f is sufficiently smooth and vanishes at $x\leqslant 0$, then formula (1.2) has an accuracy of order $O(h^2)$, otherwise it has an accuracy of order $O(h)$. On the other hand, Lubich (1985, 1986) introduced a fractional multi-step method for the Abel integral equation of first kind, and Plato (2005) considered fractional multi-step methods for weakly singular integral equations of first kind with a perturbed right-hand side. Liu and Tao (2007) solved the fractional integral equation, transforming it into an Abel integral equation of second kind. A method based on Chebyshev polynomials is given in Avazzadeh et al. (2011). Here we propose a method to solve an Abel integral equation of first kind based on a numerical approximation of fractional integrals and Caputo derivatives of a given function f belonging to $C^n[a,b]$ (see Theorem 4.2).

The structure of the paper is as follows. In Section 2 we recall the necessary definitions of fractional integrals and derivatives and explain some useful relations between them. Section 3 reviews some numerical approximations for fractional integrals and derivatives. The original results are then given in Section 4, where we introduce our method to approximate the solution of the Abel equation at the given nodes and we obtain an upper bound for the error. In Section 5 some examples are solved to illustrate the accuracy of the proposed method.

2. Definitions, relations and properties of fractional operators

Fractional calculus is a classical area with many good books available. We refer the reader to Malinowska and Torres (2012) and Podlubny (1999).

Definition 2.1. Let $\alpha>0$ with $n-1<\alpha\leqslant n, n\in\mathbb{N}$, and $a<x<b$. The left and right Riemann–Liouville fractional integrals of order α of a given function f are defined by

$${}_aJ_x^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt$$

and

$${}_xJ_b^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,$$

respectively, where Γ is Euler's gamma function, that is,

$$\Gamma(x)=\int_0^{\infty}t^{x-1}e^{-t}dt.$$

Definition 2.2. The left and right Riemann–Liouville fractional derivatives of order $\alpha>0$, $n-1<\alpha\leqslant n, n\in\mathbb{N}$, are defined by

$${}_aD_x^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x(x-t)^{n-\alpha-1}f(t)dt$$

and

$${}_xD_b^{\alpha}f(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b(t-x)^{n-\alpha-1}f(t)dt,$$

respectively.

Definition 2.3. The left and right Caputo fractional derivatives of order $\alpha>0$, $n-1<\alpha\leqslant n, n\in\mathbb{N}$, are defined by

$${}_a^CD_x^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x(x-t)^{n-\alpha-1}f^{(n)}(t)dt$$

and

$${}_x^CD_b^{\alpha}f(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b(t-x)^{n-\alpha-1}f^{(n)}(t)dt,$$

respectively.

Definition 2.4. Let $\alpha>0$. The Grunwald–Letnikov fractional derivatives are defined by

$$D^{\alpha}f(x)=\lim_{h\to 0}h^{-\alpha}\sum_{r=0}^{\infty}(-1)^{r}\binom{\alpha}{r}f(x-rh)$$

and

$$D^{-\alpha}f(x)=\lim_{h\to 0}h^{\alpha}\sum_{r=0}^{\infty}\begin{bmatrix}\alpha\\ r\end{bmatrix}f(x-rh),$$

where

$$\begin{bmatrix}\alpha\\ r\end{bmatrix}=\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+r-1)}{r!}.$$

Remark 2.5. The Caputo derivatives (Definition 2.3) have some advantages over the Riemann–Liouville derivatives (Definition 2.2). The most well known is related with the Laplace transform method for solving fractional differential equations. The Laplace transform of a Riemann–Liouville derivative leads to boundary conditions containing the limit values of the Riemann–Liouville fractional derivative at the lower terminal $x=a$. In spite of the fact that such problems can be solved analytically, there is no physical interpretation for such a type of boundary conditions. In contrast, the Laplace transform of a Caputo derivative imposes boundary conditions involving integer-order derivatives at $x=a$, which usually are acceptable physical conditions. Another advantage is that the Caputo derivative of a constant function is zero, whereas for the Riemann–Liouville it is not. For details see Sousa (2012).