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ORIGINAL ARTICLE

Exact solution of Helmholtz equation for the case of non-paraxial Gaussian beams

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Abstract A new type of exact solutions of the full 3 dimensional *spatial* Helmholtz equation for the case of non-paraxial Gaussian beams is presented here.

We consider appropriate representation of the solution for Gaussian beams *in a spherical coordinate system* by substituting it to the full 3 dimensional spatial Helmholtz equation.

Analyzing the structure of the final equation, we obtain that governing equations for the components of our solution are represented by the proper *Riccati* equations of complex value, which has no analytical solution in general case.

But we find one of the possible exact solutions which is proved to satisfy to such equations for Gaussian beams.

1. Introduction

The full 3-dimensional *spatial* Helmholtz equation provides solutions that describe the propagation of waves over space (*e.g., electromagnetic waves*) under proper boundary conditions; it should be presented in a spherical coordinate system R, θ, φ as given below (Sommerfeld, 1949; Serway, 2004):

$$\Delta A + k^2 A = 0, \tag{1.1}$$

- where Δ is the Laplacian, k is the wavenumber, and A is the amplitude. So, the derivation advanced in this manuscript starts with the scalar Helmholtz equation expressed in spherical co-ordinates.

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Besides, in spherical coordinate system (Kamke, 1971):

$$\Delta A = \frac{\partial^2 A}{\partial R^2} + \frac{2}{R}\frac{\partial A}{\partial R} + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 A}{\partial \varphi^2} + \frac{1}{R^2}\frac{\partial^2 A}{\partial \theta^2} + \frac{1}{R^2}\cot\theta\frac{\partial A}{\partial \theta}.$$

Special solutions to this equation have generated continuing interest in the optical physics community since the discovery of unusual non-diffracting waves such as Bessel and Airy beams (Alonso and Bandres, 2014a, 2014b, 2012).

Let us search for solutions of Eq. (1.1) in a *classical* form of Gaussian beams (Yi-Qing, 2013; Tagirdzhanov et al., 2011; Chen et al., 2002), which could be presented in Cartesian coordinate system as given below (Svelto, 2010):

$$A = a \cdot \frac{w_0}{w(z)} \exp\left[-\frac{x^2+y^2}{w^2(z)} - ikz - ik\frac{x^2+y^2}{2r(z)} + i\zeta(z)\right]$$

- where $w(z)$, $r(z)$, $\zeta(z)$ – are the real functions, describing appropriate parameters of a beam; $w(z)$ is the beam waist size, $r(z)$ is a wavefront radius of curvature and $\zeta\ (z)$ is the *Gouy*'s *phase shift* properly (Svelto, 2010).

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The *classical* form of Gaussian beams above could be also represented as given below

$$\exp\left[i\left(\zeta(z) - kz + i\cdot\ln w(z) + \left(\frac{i}{w^2(z)} - \frac{k}{2R(z)}\right)\cdot(x^2+y^2)\right)\right]$$
$$= \exp\left[i\left(p(z) + \frac{x^2+y^2}{2q(z)}\right)\right]$$

- where $p(z)$ is the complex phase-shift of the waves during their propagation along the z axis; $q(z)$ is the proper complex parameter of a beam, which is determining Gaussian profile of a wave in the transverse plane at position z.

Besides, let us also note that at the *left* part of the expression above we express the term $(1/w(z))$ in a form for Gaussian beams, as $\exp(i^2 \ln w(z)) = \exp(-\ln w(z))$.

The *right* part of the expression above could be transformed in a *spherical* coordinate system to the form given below:

$$A = a\cdot\exp\left[i\left(p(R,\theta) + \frac{R^2\cdot\sin^2\theta}{2q(R,\theta)}\right)\right] \qquad (*)$$

The solution (*) is additionally assumed to be independent of the azimuthal co-ordinate to observe it under well-known paraxial approximation (Svelto, 2010) also.

Then having substituted the expression (*) into Eq. (1.1), we should obtain ($\theta \neq 0$):

$$\frac{\partial^2 p(R,\theta)}{\partial R^2} + \frac{\partial^2\left(\frac{R^2}{q(R,\theta)}\right)}{\partial R^2}\frac{\sin^2\theta}{2} + i\cdot\left(\frac{\partial p(R,\theta)}{\partial R} + \frac{\partial\left(\frac{R^2}{q(R,\theta)}\right)}{\partial R}\frac{\sin^2\theta}{2}\right)^2$$
$$+\frac{2}{R}\cdot\left(\frac{\partial p(R,\theta)}{\partial R} + \frac{\partial\left(\frac{R^2}{q(R,\theta)}\right)}{\partial R}\frac{\sin^2\theta}{2}\right) + \frac{1}{R^2}\cdot\frac{\partial^2 p(R,\theta)}{\partial\theta^2}$$
$$+\frac{1}{2}\frac{\partial^2\left(\frac{\sin^2\theta}{q(R,\theta)}\right)}{\partial\theta^2} + \frac{i}{R^2}\cdot\left(\frac{\partial p(R,\theta)}{\partial\theta} + \frac{\partial\left(\frac{\sin^2\theta}{q(R,\theta)}\right)}{\partial\theta}\frac{R^2}{2}\right)^2$$
$$+\frac{\cot\theta}{R^2}\cdot\left(\frac{\partial p(R,\theta)}{\partial\theta} + \frac{\partial\left(\frac{\sin^2\theta}{q(R,\theta)}\right)}{\partial\theta}\frac{R^2}{2}\right) = i\cdot k^2. \qquad (1.2)$$

2. Exact solutions

Let us re-designate appropriate term in (*) as given below:

$$f(R,\theta) = p(R,\theta) + \frac{R^2\cdot\sin^2\theta}{2q(R,\theta)}.$$

In such a case, Eq. (1.2) could be transformed as shown below ($\theta \neq 0$):

$$\frac{\partial^2 f(R,\theta)}{\partial R^2} + i\cdot\left(\frac{\partial f(R,\theta)}{\partial R}\right)^2 + \frac{2}{R}\cdot\left(\frac{\partial f(R,\theta)}{\partial R}\right) + \frac{1}{R^2}$$
$$\cdot\left(\frac{\partial^2 f(R,\theta)}{\partial\theta^2} + i\cdot\left(\frac{\partial f(R,\theta)}{\partial\theta}\right)^2 + \cot\theta\cdot\left(\frac{\partial f(R,\theta)}{\partial\theta}\right)\right) - i\cdot k^2 = 0 \qquad (2.1)$$

Thus, all possible solutions for representing Gaussian beams in a form (*) are described by the Eq. (2.1).

But we should especially note that during the process of obtaining a solution (for example, if we are simply assuming a special eikonal solution (Svelto, 2010; Milonni and Eberly, 2010) to the Helmholtz equation), some of main features of the solution could be reduced; so, such a solution need not have any relation to Gaussian form (*).

Besides, one of the obvious solutions of PDE-equations (2.1):

$$f(R,\theta) = f1(R) + f2(\theta) \qquad (**)$$

- where $f1\ (R)$, $f2\ (\theta)$ – are the functions of *complex* value.

Let us assume as given below:

$$\frac{\partial^2 f(R,\theta)}{\partial\theta^2} + i\cdot\left(\frac{\partial f(R,\theta)}{\partial\theta}\right)^2 + \cot\theta\cdot\left(\frac{\partial f(R,\theta)}{\partial\theta}\right) = C \qquad (2.2)$$

- where C – is a constant of *complex* value. For such a case, Eq. (2.1) could be reduced as shown below ($\theta \neq 0$):

$$\frac{\partial^2 f(R,\theta)}{\partial R^2} + i\cdot\left(\frac{\partial f(R,\theta)}{\partial R}\right)^2 + \frac{2}{R}\cdot\left(\frac{\partial f(R,\theta)}{\partial R}\right) + \frac{C}{R^2} - i\cdot k^2 = 0 \qquad (2.3)$$

3. Presentation of exact solution

Under assumption (**), Eq. (2.2) could be represented as shown below:

$$\left(\frac{df_2}{d\theta}\right) = y(\theta) \Rightarrow y'(\theta) = -i\cdot y^2 - \cot\theta\cdot y + C,$$
$$y(\theta) = \csc\theta\cdot u(\theta) \Rightarrow u'(\theta) = -(i\cdot\csc\theta)\cdot u^2 + C\cdot\sin\theta, \qquad (3.1)$$

- where the last equation is known to be the *Riccati* ODE (Kamke, 1971), which has no solution in general case. But if $C = 0$, Eq. (3.1) has a proper solution ($C_0 = \text{const}$):

$$u'(\theta) = -(i\cdot\csc\theta)\cdot u^2, u(\theta) = \frac{1}{(C_0 + i\cdot\int\csc\theta d\theta)} \Rightarrow$$
$$\frac{df_2}{d\theta} = \frac{\csc\theta}{(C_0 + i\cdot\int\csc\theta d\theta)}(C_0 = 0) \Rightarrow f_2 = -i\cdot\ln(\int\csc\theta d\theta) \qquad (3.2)$$

Besides, Eq. (2.3) could be presented as given below ($C = 0$):

$$\left(\frac{df_1}{dR}\right) = y_1(R) \Rightarrow y_1'(R) = -i\cdot y_1^2 - \tfrac{2}{R}y_1 - \left(\tfrac{C}{R^2} - i\cdot k^2\right),$$
$$f_1(R) = \int y_1(R)dR. \qquad (3.3)$$

- where the last *Riccati* ODE (3.3) has a proper solution as shown below if $C = 0$ (see Kamke, 1971, the case 1.104).

Indeed, let us assume ($k \neq 0$, $R \neq 0$):

$$y_1 = u_1 + \tfrac{i}{R}, y_1'(R) = -i\cdot y_1^2 - \tfrac{2}{R}y_1 + i\cdot k^2$$
$$\Rightarrow u_1'(R) = -i\cdot u_1^2 + i\cdot k^2 \Rightarrow \int\tfrac{du_1}{k^2-u_1^2} = i\cdot R$$
$$\Rightarrow \begin{cases} u_1 = k\cdot\tanh(i\cdot k\cdot R), |i\cdot\tan(k\cdot R)| < 1, \\ u_1 = k\cdot\coth(i\cdot k\cdot R), |i\cdot\tan(k\cdot R)| > 1, \end{cases}$$

- then, we obtain:

$$\begin{cases} f_1 = -i\cdot\ln\cosh(i\cdot k\cdot R) + i\cdot\ln R, |k\cdot R| < \pi/4, \\ f_1 = -i\cdot\ln\sinh(i\cdot k\cdot R) + i\cdot\ln R, |k\cdot R| > \pi/4. \end{cases} \qquad (3.4)$$

Taking into consideration the expression (**) for the solution as well as (3.2)–(3.4), let us finally present a new type of *non-paraxial* solution, which is proved to satisfy the Helmholtz equation (1.1), as shown below: