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Exp-function method for some nonlinear PDE's and a nonlinear ODE's

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Abstract In this paper, we apply the Exp-function method to find some exact solutions for two nonlinear partial differential equations (NPDE) and a nonlinear ordinary differential equation (NODE), namely, Cahn-Hilliard equation, Allen-Cahn equation and Steady-State equation, respectively. It has been shown that the Exp-function method, with the help of symbolic computation, provides a very effective and powerful mathematical tool for solving NPDE's and NODE's. Mainly we try to present an application of Exp-function method taking to consideration rectifying a commonly occurring errors during some of recent works. The results of the other methods clearly indicate the reliability and efficiency of the used method.

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1. Introduction

The study of exact solutions of nonlinear partial differential equations (NPDE) plays an important role in mathematical physics, engineering and the other sciences. In the past several decades, various methods for obtaining solutions of NPDEs

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and ODE's have been presented, such as, tanh-function method (Wazwaz, 2005, 2006a,b), Adomian decomposition method (Hashim et al., 2006; Tatari et al., 2007), Homotopy perturbation method (Rashidi et al., 2009; Biazar et al., 2009; Berberler and Yildirim, 2009), variational iteration method (Shakeri and Dehghan, 2008; Soliman and Abdou, 2007; Yusufoglu and Bekir, 2007), spectral method (Parand and Taghavi, 2009; Parand et al., 2009, 2010), sine–cosine method (Tascan and Bekir, 2009; Wazwaz, 2007), radial basis method (Tatari and Dehghan, 2010; Dehghan and Shokri, 2009) and so on. Recently, He and Wu (2006) proposed a novel method, so called Exp-function method, which is easy, succinct and powerful to implement to nonlinear partial differential equations arising in mathematical physics. The Exp-function method has been successfully applied to many kinds of NPDEs, such as, KdV equation with variable coefficients (Zhang, 2007), Maccari's system (Zhang, 2007), Boussinesq equations (Abdou et al., 2007), Burger's equations (Ebaid, 2007; Biazar and Ayati, 2009; Ebaid, 2009), Double Sine-Gordon equation (Domairry et al., 2010; He and Abdou, 2007), Fisher equation (Ozis and

Koroglu, 2008), Jaulent–Miodek equations (He and Zhang, 2008) and the other important nonlinear partial differential equations (Koroglu and zis, 2009; Shin et al., 2009; Zhang, 2008). Recently, in some of papers applying the Exp-function method (He and Wu, 2006) have been occurred with common errors. Seven common errors are formulated and classified by (Kudryashov, 2009). In this paper we try to apply this method taking to rectifying these common errors to look exact solutions of three nonlinear differential equations, namely, Cahn-Hilliard equation, Allen-Cahn equation and Steady-State equation given by

$$u_t = \gamma u_x + 6u(u_x)^2 + (3u^2 - 1)u_{xx} - u_{xxxx},$$
$$u_t = u_{xx} - u^3 + u,$$
$$\alpha u''(x) = \beta u(x)(u(x) - m)(u(x) + m),$$

respectively, which α, β, m and γ are the constants.

The Cahn–Hilliard equation was proposed to describe phase separation phenomena in binary systems (Cahn et al., 1958). This equation is related with a number of interesting physical phenomena like the spinodal decomposition, phase separation and phase ordering dynamics. It is also very crucial in material sciences (Chan, 1961; Choo et al., 2004; Gurtin, 1996). On the other hand, this equation is very hard and difficult to solve. The Cahn-Hilliard equation has been extensively studied by Wang and Shi (1993), Jabbari and Peppas (1995), Puri and Binder (1991) for the study of interfaces. Global existence and uniqueness of the solution have been shown by Elliott and Zheng (1986). Jingxue (1992) has shown the existence of continuous solution for the problem with degenerate mobility. Recently, Dehghan and Mirzaei (2009) applied a numerical method based on the boundary integral equation and dual reciprocity methods for one-dimensional Cahn-Hilliard equation. Ugurlu and Kaya (2008) solved Cahn-Hilliard equation by tanh-function method. Furihata (2001) applied finite difference for Cahn-Hilliard equation. Many articles have investigated this equation mathematically and numerically this equation (Mello et al., 2005; Kim, 2007; Wells et al., 2006). Also, Allen-Cahn equation arise in many scientific applications such as mathematical biology, quantum mechanics and plasma physics. It is well known that wave phenomena of plasma media and fluid dynamics are modelled by kink shaped and tanh solution or bell shaped sech solutions (Wazwaz, 2007; Tascana and Bekir, 2009).

The rest of the paper is organized as follows: Section 2 describes Exp-function method for finding exact solutions to the NPDEs. The applications of the proposed analytical scheme presented in Section 3. The conclusions are discussed in the Section 4. exp-function calculations are provided in the end.

2. Basic idea of Exp-function method

We consider a general nonlinear PDE in the following form

$$N(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \ldots) = 0, \tag{1}$$

where N is a polynomial function with respect to the indicated variables or some functions which can be reduced to a polynomial function by using some transformation. We introduce a complex variation as

$$u(x, t) = U(\eta), \quad \eta = kx + \omega t, \tag{2}$$

where k and ω are constants. We can rewrite Eq. (1) in the following nonlinear ordinary differential equations

$$N(U, kU', \omega U', k^2 U'', \ldots) = 0,$$

where the prime denotes the derivation with respect to η. According to the Exp-function method (He and Wu, 2006), we assume that the solution can be expressed in the form

$$U(\eta) = \frac{\sum_{i=-d}^{c} a_i \exp(i\eta)}{\sum_{j=-q}^{p} b_j \exp(j\eta)}, \tag{3}$$

where c, d, p and q are positive integers which can be freely chosen, a_i and b_j are unknown constants to be determined. To determine the values of c and p, we balance the highest order linear term with the highest order nonlinear term in Eq. (3). Similarly to determine the values of d and q. So by means of the exp-function method, we obtain the generalized solitary solution and periodic solution for nonlinear evolution equations arising in mathematical physics.

3. Applications of the Exp-function method

Example 1. Let us consider the Cahn-Hilliard equation (Ugurlu and Kaya, 2008; Dahmani and Benbachir, 2009) in the form

$$u_t = \gamma u_x + 6u(u_x)^2 + (3u^2 - 1)u_{xx} - u_{xxxx}, \tag{4}$$

that, by using the complex variation

$$u(x, t) = U(\eta), \quad \eta = kx + \omega t, \tag{5}$$

and integrating with respect to η, Eq. (4) can be converted to the ODE (for $\gamma = 1$)

$$(\omega - k)U + k^4 U''' - 3k^2 U^2 U' + k^2 U' = 0, \tag{6}$$

where the prime denotes the derivative with respect to η and also where the integration constant is chosen as zero. In other words, we are solved this problem for the case when integration constant is zero. In view of the Exp-function method, we assume that the solution of Eq. (6) can be expressed in the form

$$U(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)},$$

where c, d, p and q are positive integers which are unknown to be determined later. In order to determine the values of c and p, we balance the linear term of the highest order with the highest order nonlinear terms in Eq. (6) i.e. U''' and $U^2 U'$. By simple calculation, we have

$$U''' = \frac{c_1 \exp[(c + 7p)\eta] + \cdots}{c_2 \exp[8p\eta] + \cdots}, \tag{7}$$

and

$$U^2 U' = \frac{c_3 \exp[(3c + p)\eta] + \cdots}{c_4 \exp[4p\eta] + \cdots} = \frac{c_3 \exp[(3c + 5p)\eta] + \cdots}{c_4 \exp[8p\eta] + \cdots}, \tag{8}$$

where c_i are determined coefficients only for simplicity. By balancing the highest order of Exp-function in Eqs. (8) and (7), we have

$$c + 7p = 3c + 5p,$$

which leads to the result

$$p = c.$$

Similarly, to determine the values of d and q, we balance the linear term of the lowest order in Eq. (6)

$$U''' = \frac{\cdots + d_1 \exp[-(7q + d)\eta]}{\cdots + d_2 \exp[-8q\eta]}, \tag{9}$$