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New exact solutions of sixth-order thin-film equation

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Abstract   The $\left(\frac{G'}{G}\right)$-expansion method is used for the first time to find traveling-wave solutions for the sixth-order thin-film equation, where related balance numbers are not the usual positive integers. New types of exact traveling-wave solutions, such as – solitary wave solutions, are obtained the sixth-order thin-film equation, when parameters are taken at special values.

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1. Introduction

Higher-order nonlinear partial differential equations have considerable attention, because of their interesting mathematical structures and surprising properties. One of the most famous examples is the sixth-order thin-film equation (Flitton and King, 2004).

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(u^n u_{xxxxx}), n > 0, \tag{1}$$

which appears in flow modeling, and describes the spread of thin viscous droplets under different driving forces. The sixth-order thin-film equation has recently become more interesting for obtaining exact analytical solutions to NLPDEs, equations from the sixth-order thin-film wave phenomena, by using appropriate techniques. Several important techniques have been developed such as the tanh-method (Krisnangkura et al., 2012), sine–cosine method (Shi et al., 2012), tanh–coth

method (Jabbari and Kheiri, 2010), exp–function method (Parand and Rad, 2012), homogeneous–balance method (Elboree, 2012a), Jacobi–elliptic function method (Honga and Lub, 2012), and first-integral method (Taghizadeh et al., 2012). All methods have limitations in their applications. In fact, no unified method can be used to handle all types of NLPDEs.

One of the most effective direct methods to develop the traveling-wave solution of NLPDEs is the $\left(\frac{G'}{G}\right)$-expansion method, which was first proposed by Wang et al. (2008). The $\left(\frac{G'}{G}\right)$-expansion method has been successfully applied to obtain the exact solution for a variety of NLPDEs (Kim and Sakthivel, 2010; Kilicman and Abazari, 2012; Ebadi et al., 2012a,b; Ayhan and Bekir, 2012; Malik et al., 2012; Elboree, 2012b; Jafari et al., 2013; Taha and Noorani, 2013; Taha et al., 2013). In this paper, the $\left(\frac{G'}{G}\right)$-expansion method is used to study the sixth-order thin-film equation in fluid mechanics for the first time. Exact traveling-wave solutions are obtained when the choice of parameters are taken at special values. Moreover, the solution obtained via this method is in good agreement with the previously obtained solutions of other researchers. Our main objective in this study is to apply the $\left(\frac{G'}{G}\right)$ method to provide the closed-form traveling-wave solutions of the sixth-order thin-film equation. To the best of our knowledge, our study is the first to apply the $\left(\frac{G'}{G}\right)$-expansion method to the sixth-order thin-film equation. In solving

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these equations, we find an instance where the related balance numbers are not the usual positive integers (see Zhang, 2009; Zayed and EL-Malky, 2011). New solitary wave solutions are also for appropriate parameters. We compare our solutions with the solutions previously obtained by Flitton and King (2004). The closed-form solution obtained via this method is in good agreement with the solutions reported in Flitton and King (2004).

Our paper is organized as follows. Section 2, provides the summary of the $\left(\frac{G'}{G}\right)$-expansion method. In Section 3, describes the applications of the $\left(\frac{G'}{G}\right)$-expansion method for the sixth-order thin-film equation. Finally, Section 4, concludes.

2. Summary of the $\left(\frac{G'}{G}\right)$-expansion method

In this section, we describe the $\left(\frac{G'}{G}\right)$-expansion method for finding the traveling-wave solutions of NLPDEs. Suppose that a nonlinear partial differential equation, in two independent variables, x and t, is given by the following

$$p(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \ldots) = 0, \tag{2}$$

where $u = u(x,t)$ is an unknown function, p is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which highest order derivatives and nonlinear terms are involved.

The summary of the $\left(\frac{G'}{G}\right)$-expansion method, can be presented in the following six steps:

Step 1:
To find the traveling-wave solutions of Eq. (2), we introduce a wave variable

$$u(x,t) = u(\zeta), \ \zeta = (x - ct), \tag{3}$$

where in the constant c is the wave velocity. By substituting Eq. (3) into Eq. (2), we obtain the following ordinary differential equations (ODEs) in ζ (which illustrate a principal advantage of a traveling-wave solution, i.e., a partial differential equation is reduced to an ODE).

$$p(u, cu', u', cu'', c^2u'', u'', \ldots) = 0. \tag{4}$$

Step 2:
If necessary, we integrate Eq. (4) as many times as possible and set the constants of integration as zero for simplicity.

Step 3:
We suppose the solution of nonlinear partial differential equation can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as the following:

$$u(\zeta) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \tag{5}$$

where $G = G(\zeta)$ satisfies the second-order linear ordinary differential equation

$$G''(\zeta) + \lambda G'(\zeta) + \mu G(\zeta) = 0, \tag{6}$$

$G' = \frac{dG}{d\zeta}$ and $G'' = \frac{d^2G}{d\zeta^2}$; a_i, λ, and μ are real constants with $a_m \neq 0$. Here, the prime denotes the derivative with respect to ζ. By using the general solutions of Eq. (6), we obtain the following expression:

$$\left(\frac{G'}{G}\right) = \begin{cases} \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \left(\frac{c_1 \sinh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta\right\} + c_2 \cosh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta\right\}}{c_1 \cosh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta\right\} + c_2 \sinh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta\right\}} \right), & \lambda^2 - 4\mu > 0, \\ \frac{-\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \left(\frac{-c_1 \sin\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta\right\} + c_2 \cos\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta\right\}}{c_1 \cos\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta\right\} + c_2 \sin\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta\right\}} \right), & \lambda^2 - 4\mu < 0, \\ \left(\frac{c_2}{c_1+c_2\zeta}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu = 0. \tag{7} \end{cases}$$

The above results can be written in simplified forms as follows:

$$\left(\frac{G'}{G}\right) = \begin{cases} \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \tanh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\zeta\right\}, & \lambda^2 - 4\mu > 0, \\ \frac{-\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \tan\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\zeta\right\}, & \lambda^2 - 4\mu < 0, \\ \left(\frac{c_2}{c_1+c_2\zeta}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu = 0. \tag{8} \end{cases}$$

Step 4:
The positive integer m can be accomplished by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (4). If we define the degree of $u(\zeta)$ as $D[u(\zeta)] = m$, the degree of other expressions is defined by the following:

$$D\left[\frac{d^q u}{d\zeta^q}\right] = m + q,$$

$$D\left[u^r \left(\frac{d^q u}{d\zeta^q}\right)^s\right] = mr + s(q + m).$$

Therefore, we obtain the value of m in Eq. (5).

Step 5:
Substitute Eq. (5) into Eq. (4), and use the general solutions of Eq. (6), and collect all terms with the same order of $\left(\frac{G'}{G}\right)$ together. Setting each coefficient of this polynomial to zero yields a set of algebraic equations for a_i, c, λ, and μ.

Step 6:
Substitute a_i, c, λ, and μ obtained in Step 5 and the general solutions of Eq. (6) into Eq. (5). Depending on the sign of the discriminant $(\lambda^2 - 4\mu)$, we can obtain the explicit solutions of Eq. (2) immediately.

3. Application of the $\left(\frac{G'}{G}\right)$-expansion method

3.1. Sixth-order thin-film equation

To find the solitary wave solution of (1), we use the following transformations.

$$u(x, t) = u(\zeta), \zeta = (x - ct). \tag{9}$$

Substituting (9) into (1) is conducted in the following:

$$(u^n u'''')' - cu' = 0. \tag{10}$$

By integrating (10) with respect to ζ and setting the integration constant equal to zero, we obtain the following:

$$u^n u'''' - cu = 0. \tag{11}$$