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ORIGINAL ARTICLE

On the numerical solution of space fractional order diffusion equation via shifted Chebyshev polynomials of the third kind

N.H. Sweilam [a], **A.M. Nagy** [b,*], **Adel A. El-Sayed** [c]

[a] *Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt*
[b] *Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt*
[c] *Department of Mathematics, Faculty of Science, Fayoum University, Fayoum, Egypt*

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Abstract In this paper, we propose a numerical scheme to solve space fractional order diffusion equation. Our scheme uses shifted Chebyshev polynomials of the third kind. The fractional differential derivatives are expressed in terms of the Caputo sense. Moreover, Chebyshev collocation method together with the finite difference method are used to reduce these types of differential equations to a system of algebraic equations which can be solved numerically. Numerical approximations performed by the proposed method are presented and compared with the results obtained by other numerical methods. The results reveal that our method is a simple and effective numerical method.

1. Introduction

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past four decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering, chemistry and other sciences (Dalir and Bashour, 2010; Kilbas et al., 2006). It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables (see for instance, Boyd (2001), Bhrawy et al. (2013, 2014a,b), Kilbas et al. (2006), Miller and Ross (1993), Oldham and Spanier (1974), Podlubny (1999), Rossikhin and Shitikova (1997)).

In recent decades, the Chebyshev polynomials are one of the most useful polynomials which are suitable in numerical analysis including polynomial approximation, integral and differential equations and spectral methods for partial differential equations and fractional order differential equations (see,

* Corresponding author.
E-mail addresses: nsweilam@sci.cu.edu.eg (N.H. Sweilam), abdelhameed_nagy@yahoo.com (A.M. Nagy), aaa18@fayoum.edu.eg (A.A. El-Sayed).

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Canuto et al., 2006; Dalir and Bashour, 2010; Mason and Handscomb, 2003; Scalas et al., 2003; Su et al., 2010; Sousa, 2011; Tadjeran et al., 2006).

In recent years, one of the attractive concepts in the initial and boundary value problems is the fractional order diffusion equation, it has found its extensive applications in many fields such as, physics, chemistry, engineering, mathematics, it included in a wide variety of practical situations and has emerged as an important area of investigation. For the general theory and applications of fractional diffusion equations see (Canuto et al., 2006; Dalir and Bashour, 2010; Oldham and Spanier, 1974; Podlubny, 1999; Scalas et al., 2003; Su et al., 2010; Sousa, 2011; Tadjeran et al., 2006). The fractional order differential equations have been much studied and many aspects of these equations are explored. For some recent work on fractional diffusion equations, we can refer to different publications (see for instance, Azizi and Loghmani (2013, 2014), Meerschaert and Tadjeran (2004, 2006), Saadatmandi and Dehghan (2007, 2006), Sweilam and Khader (2010), Sweilam et al. (2011, 2012, 2015)) and the references therein.

The fractional order (time–space) diffusion equation makes a great role in the mathematical modeling of several phenomena. It is well known that most of fractional differential equations cannot be solved exactly. Therefore, numerical methods would be proposed and investigated to get approximate solutions of these equations. The Chebyshev finite difference method and a semi-discrete scheme with Chebyshev collocation method have been introduced by Azizi and Loghmani (2013, 2014), for approximating the solution of the space fractional diffusion equations (FDEs). Also, Khader (2011) investigated the Chebyshev collocation method together with the finite difference method for solving FDEs. Moreover, Bhrawy et al. (2014b) introduced efficient generalized Laguerre-spectral methods for solving multi-term fractional differential equations on the half line. The authors in Saadatmandi and Dehghan (2006, 2011) have constructed new operational matrix and tau approach for solution of the space fractional diffusion equation. On the other side many researchers used the finite difference method (FDM) for solving FDEs (see, Elbarbary, 2003; Dehghan and Saadatmandi, 2008; Meerschaert and Tadjeran, 2004, 2006). Second kind shifted Chebyshev polynomials and Crank-Nicolson FDM are applied for solving fractional order diffusion equation in Sweilam et al. (2012, 2015).

Our fundamental goal of this work is to develop a suitable way to approximate the space fractional order diffusion equation using the shifted Chebyshev polynomials of the third kind with finite difference method together with Chebyshev collocation method. In what follows, we give some necessary definitions and mathematical relations which are used in this paper.

Definition 1. The Caputo fractional derivative operator D^{μ} of order μ is defined as the following form:

$$D^{\mu}f(x) = \frac{1}{\Gamma(m-\mu)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\mu-m+1}}\,dt, \;\; \mu > 0, \tag{1}$$

where $m-1 < \mu \leq m,\; m \in N,\; x > 0$. The linear property of the Caputo fractional derivative exists similar to the integer order differentiation:

$$D^{\mu}(\lambda f(x) + \gamma g(x)) = \lambda D^{\mu}f(x) + \gamma D^{\mu}g(x), \tag{2}$$

where λ and γ are constants.

For the Caputo derivative we can obtain the following result:

$$D^{\mu}k = 0, \quad k \text{ is a constant}, \tag{3}$$

$$D^{\mu}x^{n} = \begin{cases} 0, & \text{for } n \in \mathbb{N}_0 \text{ and } n < \lceil \mu \rceil, \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\mu)} x^{(n-\mu)}, & \text{for } n \in \mathbb{N}_0 \text{ and } n \geq \lceil \mu \rceil. \end{cases} \tag{4}$$

The function $\lceil \mu \rceil$ is used to denote the smallest integer greater than or equal to μ. Also $N_0 = \{0, 1, 2, \ldots\}$. Recall that, for $\mu \in N$ the Caputo differential operator coincides with the usual differential operator of integer order. For more details on fractional derivatives definitions, theorems and its properties see Podlubny (1999).

The main aim of this work is to find approximate solution of space fractional order diffusion equation using the shifted Chebyshev polynomials of the third kind. Consider the one-dimensional space fractional order diffusion equation of the form:

$$\frac{\partial u(x,t)}{\partial t} = p(x)\,\frac{\partial^{\mu} u(x,t)}{\partial x^{\mu}} + q(x,t), \tag{5}$$

on a finite domain $0 < x < L,\; 0 < t \leq T$ and the parameter μ refers to the fractional order of spatial derivative with $1 < \mu \leq 2$. The function $q(x,t)$ is the source term. We also assume an initial condition:

$$u(x,0) = f(x), \quad 0 < x < L, \tag{6}$$

and the boundary conditions:

$$u(0,t) = v_0(t), \quad 0 < t \leq T \tag{7}$$

$$u(L,t) = v_1(t), \quad 0 < t \leq T. \tag{8}$$

In case of $\mu = 2$, Eq. (5) is the classical second order diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = p(x)\,\frac{\partial^{2} u(x,t)}{\partial x^{2}} + q(x,t). \tag{9}$$

In this paper, we use shifted Chebyshev polynomials of third kind and recall some important properties and its analytical form. Next we use these polynomials to approximate the numerical solution of (FDE) with the aid of the Chebyshev collocation method together with the finite difference method to convert the system of equations in algebraic equations that can be solved numerically.

For this purpose, organization of paper is expressed as follows. In Section 2, we give some properties of Chebyshev polynomials of the third kind which are of fundamental importance in what follows. In Section 3, we introduce main theorem of our technique for solving space fractional order diffusion equation subject to homogeneous and nonhomogeneous boundary conditions using a shifted Chebyshev polynomials of the third kind. Numerical scheme is given in Section 4. In Section 5, we present numerical examples to exhibit the accuracy and the efficiency of our proposed method where our numerical results are computed by Matlab program. Conclusions are presented in Section 6.