Journal of King Saud University – Science (2016) **28**, 55–60

King Saud University
Journal of King Saud University – Science
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ORIGINAL ARTICLE

Study of fractional order Van der Pol equation

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Received 9 March 2015; accepted 18 April 2015
Available online 2 May 2015

KEYWORDS

Oscillator equation;
Damping;
Homotopy analysis method;
Embedding parameter;
Error analysis;
Limit cycle

Abstract In this article, Homotopy analysis method is successfully used to find the approximate solution of fractional order Van der Pol equation. The fractional derivative is described in the Caputo sense.The numerical computations of convergence control parameters for the acceleration of convergence of approximate series solution are obtained by the analysis of minimization of error for different particular cases and the results are depicted through graphs. The salient feature of the article is the graphical presentation of achieving limit cycles for different parameters.

1. Introduction

Van der Pol oscillator equation was first introduced in 1920 by Vander Pol (1920) who introduced the equation to describe the oscillation of triode in the electrical circuit. The mathematical model for this system is a second order differential equation with third degree of nonlinearity as

$$\ddot{u}(t) - \epsilon(1 - u^2(t)\dot{u}(t) + u(t) = 0, \quad (1)$$

where $\epsilon > 0$ is a control parameter and $\ddot{u}(t)$, $\dot{u}(t)$ are the second and first order derivative of u with respect to time. if $\epsilon = 0$, Eq. (1) represents the simple linear oscillator and for $\epsilon \gg 1$ it represents relaxation oscillator. The equivalent state space formulation of the Eq. (1) is

$$\frac{du_1}{dt} = u_2,$$
$$\frac{du_2}{dt} = -u_1 - \epsilon(u_1^2 - 1)u_2,$$

In the Eq. (1) for the small value of $u(t)$, the damping force is negative i.e., $-\epsilon\dot{u}(t)$. Again if $u(t)$ is bigger, it becomes dominant and the damping is positive. Van der Pol oscillator is an example of self oscillatory system which is now considered as a very useful mathematical model. Eq. (1) is also known as unforced Van der Pol equation. Van der Pol proposed another version of the above equation by including a periodic forcing term as

$$\ddot{u}(t) - \epsilon(1 - u^2(t))\dot{u}(t) + u(t) = a\sin wt \quad (2)$$

In 1945, Cartwright and Littlewood (1945) analyzed the Van der Pol equation with large nonlinearity parameter. In 1949, Levinson (1949) studied the Van der Pol equation and had shown that the equation has singular solution. The equation is considered as basic model for oscillatory process for

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2015.04.005

Physics, Biology, Electronics, and Neurology. Van der Pol himself built a number of electronic circuits to model human heart using the equation.

Many researchers have tried to solve and study the Van der Pol equation in various forms. Mickens (2001) proposed the study of a non-standard finite difference scheme for the unplugged Van der Pol equation. In 2002, Mickens (2002) studied numerically the Van der Pol equation using a non-standard finite-difference scheme . In the same year, Mickens (2002) proposed a step-size dependence of the period for a forward-Euler scheme of the Van der Pol equation. In 2003, Mickens (2003) proposed different forms of Fractional Van der Pol oscillators. Researchers have tried many methods to solve the Van der Pol differential equation using Energy balance method (Mehdipour et al., 2010; Younesian et al., 2010), Parameter expanding method (He, 2006; Xu, 2007) etc.

The Fractional order differential equations have created much interest to the researchers (Atangana and Secer, 2013) due to the non local behavior of the operator which takes into account the fact that future state depends on the present as well as on the history of the previous states. Thus fractional order derivatives are naturally related to the systems with memory which prevails for most of physical and scientific system models. Another advantage is fractional order system response ultimately converges to integer order system response. Leung et al. (2012) have used residue harmonic balance method for fractional order Van der Pol like oscillators. Gafiychuk et al. (2008) have done the analysis of fractional order Bonhoeffer Van der Pol oscillator. Leung and Guo (2011) have used forward residue harmonic balance for autonomous and non autonomous systems with fractional derivative damping. Guo et al. (2011) have given the asymptotic solution of fractional Van der Pol oscillator using the same method. Leung and Guo (2010) have used the method for discontinuous nonlinear oscillator for fractional power restoring force. Sardar et al. (2009) have found the approximate analytical solution of multi term fractionally damped Van der Pol equation. Konuralp et al. (2009) studied numerical solution of Van der Pol equation with fractional damping term. Pereira et al. (2004) have proposed a fractional order Van der Pol equation as

$$\frac{d^{\lambda}u(t)}{dt^{\lambda}} - \epsilon(1-u^2(t))\frac{du(t)}{dt} + u(t) = 0, \qquad 1 < \lambda < 2, \tag{3}$$

with the state space formulation as

$$\frac{du_1}{dt} = u_2,$$
$$\frac{d^{\lambda}u_2}{dt^{\lambda}} = -u_1 - \epsilon(u_1^2 - 1)u_2,$$

which is obtained by introducing a capacitance by a fractance in the nonlinear RLC circuit. Barbosa et al. proposed fractional order Van der Pol equation by introducing a fractional order time derivative in the state space equation of the classical Van der Pol equation as

$$\frac{d^{\lambda}u_1}{dt^{\lambda}} = u_2,$$
$$\frac{du_2}{dt} = -u_1 - \epsilon(u_1^2 - 1)u_2,$$

which gives us the Van der Pol equation as

$$\frac{d^{1+\lambda}u(t)}{dt^{\lambda}} - \epsilon(1-u^2(t))\frac{d^{\lambda}u(t)}{dt^{\lambda}} + u(t) = 0, \qquad 1 < \lambda < 2, \tag{4}$$

In the present article authors have considered the two fractional order time derivative in the state space equation as

$$\frac{d^{\alpha}u_1}{dt^{\alpha}} = u_2,$$
$$\frac{d^{\alpha}u_2}{dt^{\alpha}} = -u_1 - \epsilon(u_1^2 - 1)u_2, \quad 0 < \alpha < 1,$$

which generates the fractional order Van der Pol equation as

Table 1 Comparison of exact residual error for different values of at $\alpha = 1$.

Order of approximation	$\hbar$	E_m	E_m at $\hbar = -1$
1	−1.02178	7.23448×10^{-3}	7.56674×10^{-3}
2	−0.729311	7.59265×10^{-2}	1.09266×10^{-1}
3	−0.76059	1.07886×10^{-4}	5.11962×10^{-2}

Table 2 Comparison of exact residual error for different values of at $\alpha = 0.75$.

Order of approximation	$\hbar$	E_m	E_m at $\hbar = -1$
1	−1.04575	1.39174×10^{-2}	1.50387×10^{-2}
2	−0.624875	1.41559×10^{-1}	2.35311×10^{-1}
3	−0.758726	2.03645×10^{-4}	9.34828×10^{-2}

Table 3 Comparison of exact residual error for different values of at $\alpha = 0.5$.

Order of approximation	$\hbar$	E_m	E_m at $\hbar = -1$
1	−1.09418	8.14262×10^{-3}	1.21135×10^{-2}
2	−0.550819	1.87993×10^{-1}	3.65587×10^{-1}
3	−0.725017	2.56436×10^{-3}	1.22567

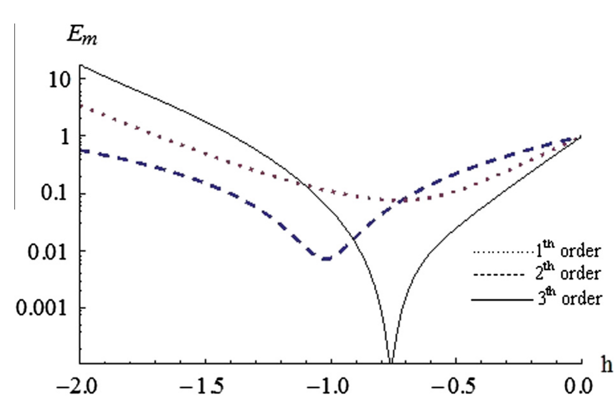

Fig. 1 Plots of exact residual error E_m versus $\hbar$ for $a = 1, \epsilon = 1$ and $\alpha = 1$.