Journal of King Saud University – Science (2016) **28**, 61–68

King Saud University
Journal of King Saud University – Science
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ORIGINAL ARTICLE

Approximate solution of integro-differential equation of fractional (arbitrary) order

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Received 6 March 2015; accepted 27 April 2015
Available online 3 June 2015

KEYWORDS
Fractional calculus;
Homotopy perturbation method;
The variational iteration method;
Fractional integro-differential equations

Abstract In the present paper, we study the integro-differential equations which are combination of differential and Fredholm–Volterra equations that have the fractional order with constant coefficients by the homotopy perturbation and the variational iteration. The fractional derivatives are described in Caputo sense. Some illustrative examples are presented.

1. Introduction

During the past decades, the topic of fractional calculus has attracted many scientists and researchers due to its applications in many areas, see Podlubny (1999), Gaul et al. (1991), Glockle and Nonnenmacher (1995); Hilfert (2000). Thus several researchers have investigated existence results for solutions to fractional differential equations due to the fact that many mathematical formulations of physical phenomena lead to integro-differential equations, for instance, mostly these types of equations arise in continuum and statistical mechanics and chemical kinetics, fluid dynamics, and biological models, for more details see Baleanu et al. (2012), Kythe and Puri (2002), Mainardi (1997).

Integro-differential equations are usually difficult to solve analytically, so it is required to obtain an efficient approximate solution. The homotopy perturbation method and variational iteration method which are proposed by He (1999a,b) are of the methods which have received much concern. These methods have been successfully applied by many authors such as Abbasbandy (2007), Abdulaziz et al. (2008); Yıldırım (2008). In this work, we study the Integro-differential equations which are combination of differential and Fredholm–Volterra equations that have the fractional order. In particular, we applied the HPM and VIM for fractional Fredholm Integro-differential equations with constant coefficients of the form

$$\sum_{k=0}^{\infty} P_k D_*^{\alpha} u(t) = g(t) + \lambda \int_0^a H(x,t)u(t)dt, \quad a \le x, \quad t \le b \tag{1.1}$$

under the initial-boundary conditions

$$\begin{aligned} D_*^{\alpha} u(a) &= u(0) \\ D_*^{\alpha} u(0) &= u'(a) \end{aligned} \tag{1.2}$$

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2015.04.006
1018-3647

where a is constant and $1 < \alpha \leq 2$ and D_*^α is the fractional derivative in the Caputo sense.

2. Preliminaries

In this section, we give some basic definitions and properties of fractional calculus theory which are used in this paper.

Definition 2.1. A real function $f(x),\ x > 0$ is said to be in space $C\mu, \mu \in R$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C(0,\infty)$, and it is said to be in the space C_μ^n if $f^n \in R_\mu,\ n \in N$.

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C\mu,\ \mu \geq -1$ is defined as:

$$J^\alpha f(x) = \frac{1}{(\Gamma(\alpha))} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, \quad t > 0 \tag{2.1}$$

in particular $J^0 f(x) = f(x)$

For $\beta \geqslant 0$ and $\gamma \geqslant -1$, some properties of the operator J^α

1. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$
2. $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$
3. $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$

Definition 2.3. The Caputo fractional derivative of $f \in C_{-1}^m,\ m \in N$ is defined as:

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^m(t)dt,\ m-1 < \alpha \leq m \tag{2.2}$$

Lemma 2.4. *if $m-1 < \alpha \leq m,\ m \in N, f \in C_\mu^m,\ \mu > -1$ then the following two properties hold*

1. $D^\alpha[J^\alpha f(x)] = f(x)$
2. $J^\alpha[D^\alpha f(x)] = f(x) - \sum_{k=1}^{m-1} f^k(0)\frac{x^k}{k!}$

3. Homotopy perturbation method

The homotopy perturbation method first proposed by He (2005, 2006) is applied to various problems (He, 2005,, 2006, 2000, 2003).

To illustrate the basic idea of this method, we consider the following nonlinear differential equation

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{3.1}$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma \tag{3.2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω.

In general, the operator A can be divided into two parts L and N, where L is linear, while N is nonlinear. Eq. (3.1) therefore can be rewritten as follows

$$L(u) + N(u) - f(r) = 0 \tag{3.3}$$

By the homotopy technique (Liao, 1995, 1997). We construct a homotopy $v(r,p) : \Omega \times [0,1] \to R$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \quad p \in [0,1],\ r \in \Omega \tag{3.4}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{3.5}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of Eq. (3.1) which satisfies the boundary conditions.

From Eqs. (3.4), (3.5) we have

$$H(v,0) = L(v) - L(u_0) = 0, \tag{3.6}$$

$$H(v,1) = A(v) - f(r) = 0. \tag{3.7}$$

The changing in the process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology this is called deformation and $L(v) - L(u_0)$, and $A(v) - f(r)$ are called homotopic.

Now, assume that the solution of Eqs. (3.4), (3.5) can be expressed as

$$v = v_0 + pv_1 + p^2 v_2 + \ldots \tag{3.8}$$

The approximate solution of Eq. (3.1) can be obtained by Setting $p = 1$.

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots \tag{3.9}$$

4. The variational iteration method

To illustrate the basic concepts of VIM, we consider the following differential equation

$$L(u) + N(u) = g(x) \tag{4.1}$$

where L is a linear operator, N nonlinear operator, and $g(x)$ is an non-homogeneous term.

According to VIM, one constructs a correction functional as follows

$$y_{n+1} = y_n + \int_0^x \lambda[Ly_n(s) - N\widetilde{y}_n(s)]ds \tag{4.2}$$

where λ is a general Lagrange multiplier, and $\widetilde{y}_n$ denotes restricted variation i.e $\delta\widetilde{y}_n = 0$.

Remark. For the analysis of HPM and VIM we refer the reader to Kadem and Kilicman (2012); Elbeleze et al. (2012).

5. Numerical examples

In this section, we have applied the homotopy perturbation method and variational iteration method to linear and nonlinear Fredholm Integro-differential equations of fractional order with known exact solution

Example 5.1. Consider the following linear Fredholm integro-differential equation: