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ORIGINAL ARTICLE

On the Perturbation–Iteration Algorithm for fractional differential equations

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Abstract In this study, Perturbation–Iteration Algorithm, namely PIA, is applied to solve some types of fractional differential equations (FDEs) for the first time. To illustrate the efficiency of the method, numerical solutions are compared with the results published in the literature by considering some FDEs. The results confirm that the PIA is a powerful, efficient and accurate method for solving nonlinear fractional differential equations.

1. Introduction

As an important mathematical branch investigating the properties of derivatives and integrals, the history of fractional calculus goes back to the beginning of the ordinary calculus. Even though there is a long period of time since the beginning of fractional calculus, various applications of fractional calculus over real engineering problems just emerged in the last decades. The evolution of the symbolic computation programs such as Mathematica, also helped this development. There is a large cycle of works as applications of fractional calculus in various branches of engineering. Especially interdisciplinary problems in applied sciences can be expressed by the fractional integrals and derivatives. In control theory (Senol et al., 2014), viscoelasticity (Yu and Lin, 1998), electrochemistry (Oldham, 2010), electromagnetic (Heaviside, 2008), fractional differential equations are used for modeling some different types of engineering problems. For example, in Hamamci (2007). Some basic definitions and applications of fractional derivatives are also given in Mainardi (1997) and Podlubny (1998).

As parallel to the studies in engineering and science, much attention is given to FDEs in applied mathematics. Especially, solution of FDE (analytically or/and numerically) methods is studied by many author in the last decades.

Existence and uniqueness of the solutions are also studied by some mathematicians, (see Yakar and Koksal (2012) and the references therein).

There are a number of differential equations whose solution can not be found analytically. Those situations appear in FDEs more than other types of differential equations. In this case, as the study of algorithms using numerical approximation for the problems of mathematical analysis, the field of numerical analysis is used for approximate solutions of FDEs.

Various numerical methods have been studied for approximate solutions of FDEs. These methods include, fractional

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variational iteration method (Wu and Lee, 2010; Guo and Mei, 2011; Wu and Baleanu, 2013), homotopy perturbation method (Abdulaziz et al., 2008; He, 2012, Momani and Odibat, 2007; Zhang et al., 2014), Adomian decomposition method (Duan et al., 2012) and fractional differential transform method (Momani et al., 2007; Arikoglu and Ozkol, 2009; El-Sayed et al., 2014).

In this paper, the previously developed method (PIA), suitable for a broad class of equations which do not require special assumptions and transformations , is studied to give approximate solutions of some fractional differential equations.

Two major definitions of fractional derivative of an arbitrary order are the Riemann–Liouville and Caputo fractional derivatives. The two definitions have different orders of evaluation of derivation.

Riemann–Liouville fractional integration of order α is defined by:

$$J^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt, \quad \alpha>0, \quad x>0. \tag{1}$$

The following two equations are defined as Riemann–Liouville and Caputo fractional derivatives of order α, respectively.

$$D^{\alpha}f(x)=\frac{d^m}{dx^m}(J^{m-\alpha}f(x)), \tag{2}$$

$$D_*^{\alpha}f(x)=J^{m-\alpha}\left(\frac{d^m}{dx^m}f(x)\right), \tag{3}$$

where $m-1<\alpha<m$ and $m\in\mathbf{N}$.

Due to the appropriateness of the initial conditions, fractional definition of Caputo is often used in recent years.

Definition 1.1. The fractional derivative of $u(x)$ in the Caputo sense is defined as

$$D_*^{\alpha}u(x)=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}u^{(m)}(t)dt, & m-1<\alpha<m,\\ \frac{d^m}{dx^m}, & \alpha=m\end{cases} \tag{4}$$

for $m-1<\alpha<m,\ m\in\mathbf{N},\ x>0,\ u\in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 1.1. *If $m-1<\alpha<m, m\in\mathbf{N}$ and $u\in C_{\mu}^m, \mu-1$ then*

$$D_*^{\alpha}J^{\alpha}u(x)=u(x) \tag{5}$$

and

$$J^{\alpha}D_*^{\alpha}u(x)=u(x)-\sum_{k=0}^{m-1}u^{(k)}(0^+)\frac{x^k}{k!}, \quad x>0. \tag{6}$$

After this introductory section, Section 2 is reserved for a brief review of the Perturbation–Iteration Algorithm PIA (1,1), in Section 3 some examples are presented to show the efficiency and simplicity of the algorithm. Finally the paper ends with a conclusion in Section 4.

2. Overview of the Perturbation–Iteration Algorithm PIA(1,1)

As one of the most practical subjects of physics and mathematics, differential equations create models for a number of problems in science and engineering to give an explanation for a better understanding of the events. Perturbation methods have been used for this purpose for over a century (Nayfeh, 2008; Jordan and Smith, 1987; Skorokhod et al., 2002). These methods could be used to search approximate solutions of integral equations, difference equations, integro-differential equations and partial differential equations.

But the main difficulty in the application of perturbation methods is the requirement of a small parameter or to install a small artificial parameter in the equation. For this reason, the obtained solutions are restricted by a validity range of physical parameters. Therefore, to overcome the disadvantages come with the perturbation techniques, some methods have been suggested by several authors (He, 2001; Mickens, 1987, 2005, 2006; Cooper and Mickens, 2002; Hu and Xiong, 2003; He, 2012; Wang and He, 2008; Iqbal and Javed, 2011; Iqbal et al., 2010).

Parallel to these studies, recently a new Perturbation–Iteration Algorithm has been proposed by Aksoy, Pakdemirli and their co-workers (Aksoy and Pakdemirli, 2010; Pakdemirli et al., 2011; Aksoy et al., 2012). A previous attempt of constructing root finding algorithms systematically (Pakdemirli and Boyac, 2007; Pakdemirli et al., 2007, 2008) finally guided to generalization of the method to differential equations also (Aksoy and Pakdemirli, 2010; Pakdemirli et al., 2011; Aksoy et al., 2012). In the new technique, an iterative algorithm is established on the perturbation expansion. The method has been applied to first order equations (Pakdemirli et al., 2011) and Bratu type second order equations (Aksoy and Pakdemirli, 2010) to obtain approximate solutions. Then the algorithms were tested on some nonlinear heat equations also (Aksoy et al., 2012). Finally, the solutions of the Volterra and Fredholm type integral equations (Dolapci et al., 2013) and ordinary differential equation and systems (Şenol et al., 2013) are given by the present method.

In this study, the previously developed new technique is applied to some types of nonlinear fractional differential equations for the first time. To obtain the approximate solutions of equations, the most basic Perturbation–Iteration Algorithm PIA(1,1) is employed by taking one correction term in the perturbation expansion and correction terms of only first derivatives in the Taylor series expansion, i.e. $n=1, m=1$.

Consider the following initial value problem.

$$D^{\alpha}u(t)+N[u(t)]+L[u(t)]=g(t), \quad \alpha>0, \tag{7}$$

$$u^{(k)}(0)=c_k, \quad k=0,1,2,\ldots,m-1, \quad m-1<\alpha\leqslant m, \tag{8}$$

where L is a linear and N is a nonlinear operator and D^{α} is the Caputo fractional derivative of order α, which is defined by:

$$D^{\alpha}u(t)=J^{m-\alpha}\left(\frac{d^m}{dt^m u(t)}\right), \quad m-1<\alpha<m,\ m\in\mathbf{N}. \tag{9}$$

In this method as ε is the artificially introduced perturbation parameter we use only one correction term in the perturbation expansion.

$$\begin{aligned}u_{n+1}&=u_n+\varepsilon(u_c)_n,\\ u'_{n+1}&=u'_n+\varepsilon(u'_c)_n.\end{aligned} \tag{10}$$

Replacing (10) into (7) and writing in the Taylor Series expansion for only first order derivatives gives