Journal of King Saud University – Science (2016) **28**, 99–102

King Saud University
Journal of King Saud University – Science
www.ksu.edu.sa
www.sciencedirect.com

ORIGINAL ARTICLE

Mittag-Leffler function for discrete fractional modelling

Guo-Cheng Wu [a,b], **Dumitru Baleanu** [c,d,*], **Sheng-Da Zeng** [a,b], **Wei-Hua Luo** [a,b]

[a] *Data Recovery Key Laboratory of Sichuan Province, College of Information Science and Mathematics, Neijiang Normal University, Neijiang 641100, Sichuan, China*
[b] *Institute of Applied Nonlinear Science, College of Mathematics and Information Science, Neijiang Normal University, Neijiang 641100, Sichuan, China*
[c] *Department of Mathematics and Computer Sciences, Cankaya University, 06530 Balgat, Ankara, Turkey*
[d] *Institute of Space Sciences, Magurele-Bucharest, Romania*

Received 1 April 2015; accepted 15 June 2015
Available online 27 June 2015

KEYWORDS

Discrete fractional calculus;
Discrete Mittag Leffler function

Abstract From the difference equations on discrete time scales, this paper numerically investigates one discrete fractional difference equation in the Caputo delta's sense which has an explicit solution in form of the discrete Mittag-Leffler function. The exact numerical values of the solutions are given in comparison with the truncated Mittag-Leffler function.

1. Introduction

The fractional calculus has become as an efficient tool in various applied areas. For example, due to the beautiful memory effects, it often appears in diffusion in the porous media (Benson et al., 2000; Berkowitz et al., 2002; Bhrawya and Zaky, 2015; Liu et al., 2004; Sun et al., 2009; Yang et al., 2013), the material's properties (Bagley and Torvik, 1983; Carpinteri and Cornetti, 2002; Mainardi, 2010; Rossikhin and Shitikova, 1997), biological population (Atangana, 2014) and control systems (Baleanu et al., 2011; Li and Chen, 2004; Machado, 1997) et al.

In the applications of the mentioned topic, one frequently comes across the discrete Mittag-Leffler function (DMLF) (Abdeljawad, 2011; Acar and Atici, 2013; Atici and Eloe, 2007; Pillai and Jayakumar, 1995; Nagai, 2003; Liu et al., 2014). Due to the functions' infinity series' expression, the truncated form is often approximately used. However, the function leads to truncated errors in fractional modelling and explicit analytical calculus for real-world applications. As a result, much effort has been dedicated to numerical approximations of the Mittag-Leffler functions (Garrappa and Popolizio, 2013; Moret and Novati, 2011; Podlubny, 2005; Valerio and Machado, 2014).

Very recently, the discrete fractional calculus is developed as a discrete fractional modelling tool (Abdeljawad, 2011;

* Corresponding author.
E-mail addresses: wuguocheng@gmail.com (G.-C. Wu), dumitru@cankaya.edu.tr (D. Baleanu), zengshengda@163.com (S.-D. Zeng), huaweiluo2012@163.com (W.-H. Luo).
Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2015.06.004
1018-3647

Atici and Eloe, 2009, 2011; Chen et al., 2011). In this paper, we consider exact numerical values of the DMLF (Abdeljawad, 2011; Abdeljawad et al., 2012; Acar and Atici, 2013) which was given in a limit

$$E_v(\lambda,t)=\lim_{k\to\infty}\sum_{i=0}^{k}\lambda^i\frac{(t+i(v-1))^{(iv)}}{\Gamma(1+iv)},\qquad(1)$$

where $t^{(v)}$ is the falling factorial function as

$$t^{(v)}=\frac{\Gamma(t+1)}{\Gamma(t+1-v)}.$$

The numerical formula of one fractional difference equation is given by the discrete fractional difference in this paper. It is organised as follows. In Section 2, some preliminaries of the fractional calculus are introduced and the numerical formula is given by the DFC. In Section 3, the DMLF is illustrated through the solution of the discrete fractional difference equation. In Section 4, we conclude our work.

2. Preliminaries

Let's revisit basics of the fractional sum and difference in the following.

Definition 2.1 (*See Atici and Eloe, 2009, 2011*). Let $\phi(t):\mathbb{N}_a=\{a,a+1,a+2,\ldots,\}\to\mathbb{R}$ and $0<v$. Then, the fractional v order sum is defined by

$${}_a\Delta_t^{-v}\phi(t)=\frac{1}{\Gamma(v)}\sum_{s=a}^{t-v}(t-s-1)^{(v-1)}\phi(s),\quad t\in(\mathbb{N})_{a+v},\qquad(2)$$

where $a\in\mathbb{R}$.

Definition 2.2 (*See Abdeljawad, 2011*). For $m-1<v\leqslant m$, and $\phi(t)$ defined on $\mathbb{N}_a$, the left Caputo-like delta difference is defined by

$${}_a^C\Delta_t^v\phi(t)=\frac{1}{\Gamma(m-v)}\sum\nolimits_{s=a}^{t-(m-v)}(t-s-1)^{(m-1-v)}\Delta^m\phi(s),$$
$$t\in\mathbb{N}_{a+(m-v)}.\qquad(3)$$

Within this definition, existence results of fractional difference equation are investigated in Chen et al. (2011)

$${}_a^C\Delta_t^v u(t)=f(u(t+v-1),\ t+v-1),$$
$$t\in\mathbb{N}_{a+1-v},u(a)=\Omega,\ 0<v\leq1,\qquad(4)$$

among which a discrete equivalent form was given as

$$u(t)=\Omega+\frac{1}{\Gamma(v)}\sum_{s=a+1-v}^{t-v}(t-s-1)^{(v-1)}f(u(s+v-1),s+v-1),$$
$$t\in\mathbb{N}_{a+1}.\qquad(5)$$

We can have a numerical formula as

$$u(a+n)=u(a)+\frac{1}{\Gamma(v)}\sum_{j=1}^{n}\frac{\Gamma(n-j+v)}{\Gamma(n-j+1)}f(u(a+j-1),$$
$$a+j-1),\quad u(a)=\Omega.\qquad(6)$$

Different from fractional differential equations or differential ones, the iteration solution from (6) is an exact one of the difference equation. In this way, the discrete fractional maps have been proposed as well as the chaotic behaviours and diffusion discussed (Wu and Baleanu, 2014a,b; Wu et al., 2015).

3. Discrete fractional difference equation

Let us consider one example about the DMLF. The following fractional difference equation

$${}_a^C\Delta_t^v y(t)=\lambda y(t+v-1),\quad y(a)=y_0,\ t\in\mathbb{N}_{a+1-v},$$
$$a=0,\ 0<v\leqslant1\qquad(7)$$

has an exact solution,

$$y(t)=y_0E_\alpha(\lambda,t),\quad t\in\mathbb{N}_0.\qquad(8)$$

Considering the existence results (Chen et al., 2011), one can have the sufficient condition

$$-\frac{1}{1+v}<\lambda<\frac{1}{1+v}.\qquad(9)$$

From the fractional sum (6), we can have

$$y(n)=y_0+\frac{\lambda}{\Gamma(v)}\sum_{j=1}^{n}\frac{\Gamma(n-j+v)}{\Gamma(n-j+1)}y(j-1),\quad y_0=1\qquad(10)$$

with which we plot a numerical solution for $v=0.8$ in Fig. 1.

In Fig. 2, we compare the exact numerical solution with the truncated DMLFs

$$y_3=\sum_{i=0}^{3}\lambda^i\frac{(t+i(v-1))^{(iv)}}{\Gamma(1+iv)}\qquad(11)$$

and

$$y_4=\sum_{i=0}^{4}\lambda^i\frac{(t+i(v-1))^{(iv)}}{\Gamma(1+iv)},\quad t\in\mathbb{N}_0.\qquad(12)$$

We can conclude that for $i\to\infty, y_i$ tends to $E_a(\lambda,t)$.

From the above comparison, we suggest that we can directly use the fractional difference equation in fitting

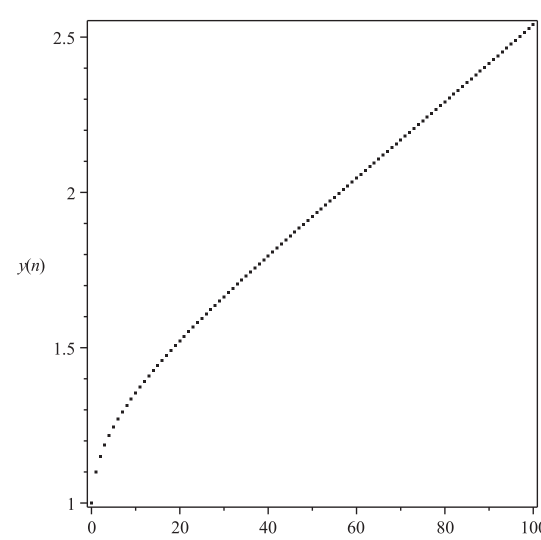

Figure 1 Exact numerical solution for $\lambda=0.1, v=0.8$ and $y_0=1$.