Journal of King Saud University – Science (2016) **28**, 103–110

King Saud University
Journal of King Saud University – Science
www.ksu.edu.sa
www.sciencedirect.com

ORIGINAL ARTICLE

On the existence and uniqueness of solutions for a class of non-linear fractional boundary value problems

Arwa Ba Abdulla, Mohammed Al-Refai *, Ahmed Al-Rawashdeh

Department of Mathematical Sciences, UAE University, United Arab Emirates

Received 18 April 2015; accepted 7 May 2015
Available online 22 May 2015

KEYWORDS

Fractional differential equations;
Boundary value problems;
Maximum principle;
Lower and upper solutions;
Caputo fractional derivative

Abstract In this paper, we extend the maximum principle and the method of upper and lower solutions to study a class of nonlinear fractional boundary value problems with the Caputo fractional derivative $1 < \delta < 2$. We first transform the problem to an equivalent system of equations, including integer and fractional derivatives. We then implement the method of upper and lower solutions to establish existence and uniqueness results to the resulting system. At the end, some examples are presented to illustrate the validity of our results.

1. Introduction

Differential equations with fractional order are generalization of ordinary differential equations to non-integer order. In recent years, a great interest was devoted to study fractional differential equations, because of their appearance in various applications in Engineering and Physical Sciences, see Hilfer (2000), Luchko (2013), Mainardi (2010), Yang (2012), Yang and Baleanu (2013). Therefore, numerical and analytical techniques have been developed to deal with fractional differential equations (Agarwal et al., 2014; Al-Refai et al., 2014; Bhrawy and Zaky, 2015a,b; Nyamoradi et al., 2014; Li et al., 2011; Yang et al., 2013). The maximum principle and the method of lower and upper solutions are well established for differential equations of elliptic, parabolic and hyperbolic types (Pao, 1992; Protter and Weinberger, 1984). Recently, there are several studies devoted to extend, if possible, these results for fractional differential equations (Agarwal et al., 2010; Al-Refai and Hajji, 2011; Al-Refai, 2012; Furati and Kirane, 2008; Lakshmikantham and Vatsala, 2008; Luchko, 2009). It is noted that the extension is not a straightforward process, due to the difficulties in the definition and the rules of fractional derivatives. Therefore, the theory of fractional differential equations is not established yet and there are still many open problems in this area. Unlike, the integer derivative, there are several definitions of the fractional derivative, which are not equivalent in general. However, the most popular ones are the Caputo and Riemann–Liouville fractional derivatives. In this paper, we prove the existence and uniqueness of solutions to the fractional boundary value problem

* Corresponding author.
E-mail addresses: arwab@uaeu.ac.ae (A.B. Abdulla), m_alrefai@uaeu.ac.ae (M. Al-Refai), aalrawashdeh@uaeu.ac.ae (A. Al-Rawashdeh).
Peer review under responsibility of King Saud University.

ELSEVIER **Production and hosting by Elsevier**

http://dx.doi.org/10.1016/j.jksus.2015.05.001
1018-3647

$$D_{0^+}^{\delta} y + f(t, y, y') = 0, \quad 0 < t < 1,\ 1 < \delta < 2, \tag{1.1}$$

$$y(0) = a,\ y'(1) = b, \tag{1.2}$$

where f is continuous with respect to t on $[0, 1]$ and smooth with respect to y and y', and the fractional derivative is considered in the Caputo's sense. Several existence and uniqueness results for various classes of fractional differential equations have been established using the method of lower and upper solutions and fixed points theorems. The problem (1.1) with $f = f(t, y)$ and non-homogenous boundary conditions of Dirichlet type was studied by Al-Refai and Hajji (2011), where some existence and uniqueness results were established using the monotone iterative sequences of upper and lower solutions. In addition, the same problem (1.1) with $f(t, y) = f_0(t, y) + f_1(t, y) + f_2(t, y)$ was studied by Hu et al. (2013) using quasi-lower and quasi-upper solutions and monotone iterative technique. The problem (1.1) with $f = f(t, y)$ and homogeneous boundary conditions of Dirichlet type and $D_{0^+}^{\delta}$ is the standard Riemann–Liouville fractional derivative discussed by Bai and Lu (2005). They used certain fixed point theorems to establish the existence and multiplicity of positive solutions for the problem.

To the best of our knowledge, the method of monotone iterative sequences of lower and upper solutions has not been implemented for the problem (1.1)–(1.2), where the nonlinear term $f = f(t, y, y')$ depends on the variables y and y'. In order to apply the method of lower and upper solutions, we need some information about the fractional derivative of a function at its extreme points. While some estimates were obtained by Al-Refai (2012) for the fractional derivative $1 < \delta < 2$, these estimates require more information about the function, unlike the case when $0 < \delta < 1$. Therefore, we transform the problem (1.1)–(1.2) to an equivalent system of two equations and then we apply the method of lower and upper solutions to the new system.

This paper is organized as follows. In Section 2, we present some basic definitions and preliminary results. In Section 3, we establish the existence and uniqueness of solutions for an associated linear system of fractional equations using the Banch fixed point theorem. In Section 4, we establish the existence and uniqueness of maximal and minimal solution to the problem. Some illustrated examples are presented in Section 5. Finally, in Section 6, we present some concluding remarks.

2. Preliminary results

The left Caputo fractional derivative of order $\alpha > 0$, for $n - 1 < \alpha < n$, $n \in \mathbb{N}$ of a function f is defined by

$$(D_{0^+}^{\alpha} f)(t) = \left(I_{0^+}^{n-\alpha} \frac{d^n}{dt^n} f\right)(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds, & n-1 < \alpha < n \in \mathbb{N}, \\ f^{(n)}(t), & \alpha = n \in \mathbb{N}, \end{cases}$$

where Γ is the well-known Gamma function and $I_{0^+}^{\alpha}$ is the left Riemann–Liouville fractional integral defined by

$$(I_{0^+}^{\alpha} f)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{2.1}$$

For more details about the definition and properties of fractional derivatives, the reader is referred to Ortigueira (2011), Podlubny (1993). In the following, we transform the problem (1.1)–(1.2) to a system of differential equations, consisting of a fractional derivative and an integer derivative. Let $y_1 = y$, and $y_2 = y\prime = Dy$. Using the fact that $D_{0^+}^{\delta} y = D_{0^+}^{\delta-1}(Dy)$ for $1 < \delta < 2$, the system (1.1)–(1.2) is reduced to

$$Dy_1 - y_2 = 0,\ \ 0 < t < 1, \tag{2.2}$$

$$D_{0^+}^{\alpha} y_2 + f(t, y_1, y_2) = 0,\ \ 0 < t < 1,\ 0 < \alpha < 1, \tag{2.3}$$

$$y_1(0) = a,\ y_2(1) = b, \tag{2.4}$$

where $\alpha = \delta - 1$. For the above system we initially require that $y_1, y_2 \in C^1[0, 1]$ and f is continuous with respect to the variable t and smooth with respect to the variables y_1 and y_2.

We have the following definition of lower and upper solutions for the system (2.2)–(2.4).

Definition 2.1 (*Lower and Upper Solutions*). A pair of functions $(v_1, v_2) \in C^1[0, 1] \times C^1[0, 1]$ is called a pair of lower solutions of the problem (2.2)–(2.4), if they satisfy the following inequalities

$$Dv_1 - v_2 \le 0, \quad 0 < t < 1, \tag{2.5}$$

$$D_{0^+}^{\alpha} v_2 + f(t, v_1, v_2) \le 0, \quad 0 < t < 1,\ 0 < \alpha < 1, \tag{2.6}$$

$$v_1(0) \le a, \quad v_2(1) \le b. \tag{2.7}$$

Analogously, a pair of functions $(w_1, w_2) \in C^1[0, 1] \times C^1[0, 1]$ is called a pair of upper solutions of the problem (2.2)–(2.4), if they satisfy the reversed inequalities. In addition, if $v_1(t) \le w_1(t)$ and $v_2(t) \le w_2(t), \forall t \in [0, 1]$, we say that (v_1, v_2) and (w_1, w_2) are ordered pairs of lower and upper solutions.

The following important results will be used throughout the text.

Lemma 2.1. Al-Refai (2012). *Let $f \in C^1[0, 1]$ attain its absolute minimum at $t_0 \in (0, 1]$, then*

$$(D_{0^+}^{\alpha} f)(t_0) \le \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)} [f(t_0) - f(0)] \le 0, \text{ for all } 0 < \alpha < 1.$$

Lemma 2.2. Changpin and Weihua (2007). *If $f \in C^n[0, 1]$ and $n - 1 < \alpha < n \in Z^+$, then $(D_{0^+}^{\alpha} f)(0) = 0$.*

We have the following new positivity result.

Lemma 2.3 (*Positivity Result*). *Let $\omega(t)$ be in $C^1[0, 1]$ that satisfies the fractional inequality*

$$D_{0^+}^{\alpha} \omega(t) + \mu(t)\omega(t) \ge 0,\ 0 < t < 1,\ 0 < \alpha < 1, \tag{2.8}$$

where $\mu(t) \ge 0$ and $\mu(0) \neq 0$. Then $\omega(t) \ge 0,\ \forall t \in [0, 1]$.

Proof. Assume by contradiction that $\omega(t) < 0$, for some $t \in [0, 1]$. As $\omega(t)$ is continuous on $[0, 1]$, $\omega(t)$ attains an absolute minimum value at $t_0 \in [0, 1]$ with $\omega(t_0) < 0$. If $t_0 \in (0, 1]$, then by Lemma 2.1, we have

$$\Gamma(1-\alpha)(D_{0^+}^{\alpha}\omega)(t_0) \le t_0^{-\alpha}[\omega(t_0) - \omega(0)] < 0.$$

Since $\Gamma(1-\alpha) > 0$, for $0 < \alpha < 1$, we have $(D_{0^+}^{\alpha}\omega)(t_0) < 0$, and hence