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Optimal control of a stochastic production-inventory model with deteriorating items

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Abstract This paper considers a stochastic optimal control of an inventory model with a deterministic rate of deteriorating items. The dynamics of the inventory model includes a perturbation by a Wiener process. The paper uses Hamilton–Jacobi–Bellman principle to find a nonlinear partial differential equation that the value function must satisfy. The partial differential equation is solved by assuming a particular form for the solution and finding three functions $Q(t)$, $M(t)$, and $R(t)$ of time by substituting the assumed solution form back in the partial differential equation. The paper then proceeds to find the optimal expected production rate and the optimal expected inventory level. The paper discusses some special cases for specific parameter values and provides some numerical examples.

1. Introduction

Including stochastic elements in planning problems is more realistic than simple deterministic analysis. Therefore, the framework used in this paper, the stochastic optimal control, are well suited for addressing many general problems including inventory control problems. The literature on the inventory models with deteriorating items has been around for decades, leading to survey papers such as Goyal and Giri (2001). There have also been papers that combine the idea of deteriorating inventory items with optimal control such as Benkherouf and Aggoun (2002). In this paper, we study a stochastic optimal control of a production-inventory model with deteriorating items in the presence of random disturbances.

Stochastic control problems seem to be a simple idea of nudging a stochastic process in one direction or another as it flows through time. However, the developments in this field have been multifaceted and it is therefore useful to have a framework for the discussion. The framework to be used here is outlined in Kendrick (2005) and Kendrick and Amman (2006), which provides a classification system for the stochastic control models that are used in economics.

There are two principal attributes used in Sethi and Thompson (2000) to classify economics stochastic control models, which are the stochastic elements and the solution method. Models with no stochastic elements are called deterministic.

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Among the models with stochastic elements, the least complexes are those with a single uncertain vector, namely additive noise terms. More complicated models which have uncertain parameters can be found in Kendrick (2005), such as measurement errors, uncertain initial state vectors, and time-varying parameters.

In a deterministic inventory system, it is assumed that the values of the state variable can be measured. In many cases, the assumption that the value of a state variable can be directly measured and exactly determined may be not realistic. In fact, the stochastic description of the production planning model is more realistic than deterministic description (Sethi and Thompson, 2000).

In the stochastic optimal control theory, the state of the system is represented by a controlled stochastic process. Therefore, the involvement of time in the state of the system will be described as a stochastic differential equation. In this paper we will only consider a stochastic differential equation of a type known as Itô equation. This equation arises when the state equation is perturbed by a Markov diffusion process (Davis, 1994; El-Gohary, 2005; Yin et al., 2003).

A related model was discussed by Presman and Sethi (2006) where the demand process is made up of a continuous part and a compound Poisson process. They have shown that the (s, S) policy is optimal by using an appropriate potential function. This function is then shown to satisfy the dynamic programing associated with the problem. Also, Benkherouf and Johnson (2009) examined the stochastic single item continuous review inventory model with a fixed ordering cost and where the demand is driven by a special type of a piecewise Markov deterministic process.

In this paper, we will be concerned with a stochastic production-inventory model with deteriorating items. We initially mention a related stochastic model which has been treated in Sethi and Thompson (2000), which can be derived as a special case of the model we study in this paper. Such a model can be applied to a system subjected to random disturbances. For instance, sales may follow a stochastic process which affects the deterministic inventory model.

The problem of production-inventory planning is one of the Operations Research and Management Science problems that have received a considerable amount of attention. Applications on the optimal production-inventory planning are also widely reported in the literature, see for example (Kenné and Gharbi, 2004; Parlar, 1985; Perkins and Kumar, 1994; Yang et al., 1999).

Shen (1994) argues that uncertainty in a stochastic control model could be classified into three categories: the first is system uncertainty, the second is parameter uncertainty, and the third is measurement uncertainty. In this paper, we assume a system uncertainty and model this by adding a random error term to the system state equation. The optimal production rate and optimal inventory level will be discussed. The problem is to find the expected optimal production rate over the planning horizon. In Section 3, the general solution of the optimal control problem will be derived. In Section 4, illustration examples and numerical examples are presented for different cases of demand rates.

2. The stochastic production-inventory model

We consider a controlled dynamic system that is affected by random noises. We aim to find strategies that minimize the expected cost over a finite time horizon while satisfying a number of constraints.

In this section, we will construct and then solve the stochastic optimal control of a production-inventory model with deterministic deteriorating items. We will introduce the possibility of controlling a system governed by Itô stochastic differential equation. We then obtain optimal states estimation for a linear system with noise.

2.1. Stochastic optimal control

This subsection is devoted to the model assumptions. We introduce the mathematical description of the problem of stochastic production-inventory system with deteriorating items. Consider a factory producing a homogeneous products that it stores in an inventory warehouse. The problem statement and notation will be introduced. Let us define the following quantities:

$X(t)$	stochastic inventory level at time t (state variable)
$U(t)$	stochastic production rate at time t (control variable)
$S(t)$	demand rate at time t (exogenous function)
T	length of the planning period
x_1	factory-inventory goal level
u_1	factory-production goal rate
x_0	initial inventory level
h	inventory holding cost coefficient
c	production cost coefficient
B	salvage value per unit of the inventory at time T
$z(t)$	standard Wiener process
σ	diffusion coefficient
θ	the deterioration coefficient

The stochastic state equation of this model can be expressed as the Itô stochastic differential equation

$$dX(t) = [U(t) - S(t) - \theta X(t)]dt + \sigma dz(t), \quad X(0) = x_0, \tag{2.1}$$

where x_0 is the initial inventory level. The Wiener process z_t can be expressed as $w(t)dt$ where $w(t)$ is a white noise process (Kendrick and Amman, 2006). Now the problem is to find the optimal production rate U_t that minimizes the expected total cost. In other words, we need to find the control function U_t that maximizes the expected revenue

$$\max_{U(t)} E\left\{\int_0^T \left[-c(U(t)-u_1)^2 - h(X(t)-x_1)^2\right]dt + BX_T\right\}, \tag{2.2}$$

It can be shown that the integrand of the objective function (2.2) is a negative definite form. The parameters x_1 and u_1 that represent the factory-inventory goal level and factory-production goal rate can be selected by the firm.

In this study, we do not restrict our attention to non-negative production rate as required in the deterministic system. Therefore, we do not restrict production rate to be non-negative. The solution of this problem will be carried out via the Hamilton–Jacobi–Bellman principle for stochastic differential equation of the Itô type.

2.2. General solution of the problem

In this subsection, we obtain the optimal production rate that minimizes the holding and production costs. Hamilton–Jacobi equation for a certain value function which represents the