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ORIGINAL ARTICLE

Flat-ended circular cylindrical punch for initially stressed Neo-Hookean solids

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Abstract The present paper reports the indentation of integral transform technique for a semi-infinite initially stressed elastic medium under the action of an axi-symmetric flat-ended circular cylindrical punch pressing the medium normally. The incremental deformation theory is used to solve the problems for Neo-Hookean solid. The distribution of incremental stress and strain is obtained by using the Hankel's transformation. The effects of the punch have been studied numerically and presented in various forms of curves. The plane punch indentation has its broad applications in the field of Engineering Mechanics. There are so many firing and launching pads, which use the Neo-Hookean solid as buffer and bear the punch during the action of machines. Thus the present problem has a lot of applications to find the effect of punch on machines.

1. Introduction

The incremental deformation theory of elasticity concerns a deformation when the state of strain and stress at any time differ only slightly from that of a known finite deformation. Various elastic bodies possess initial stress by the action of body forces. If such a body is further subjected to deforming forces then apart from the initial finite deformation, it will have incremental deformation. The basic equations of this theory have been given by Green et al. (1952), Green and Zerna (1954), Biot (1938, 1940), Neuber (1943), Mushkhelishvili (1953) and Trefftz (1933).

Biot has discussed some interesting problems with the help of this theory like surface buckling, internal buckling, etc. He used the Cartesian concepts and Elementary Mathematical Model in place of tensor calculus. Therefore, Biot's concepts are not only easy to understand the physical meaning of incremental stress and strain but also useful in mathematical analysis.

Later on, Kurashige (1969, 1971) discussed a circular crack problem and the two dimensional crack problems for initially stressed Neo-Hookean solids. Hara et al. (1989) has studied an axi-symmetric contact problem of a transversely isotropic layer indented by an annular rigid punch. Some contact problems

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Nomenclature

x_i	cartesian coordinates	w_{ii}	incremental rotation
S_{ij}	initial stress, corresponding to initial finite deformation, referred to x_i	μ_0	shear modulus in an unstrained state
n_i	components of unit normal to boundary surface	E	incremental volume expansion
ρ	density in a finite deformation	P	initial all-around compressive stress
u_i	incremental displacement (infinitesimal)	p_0	pressure in a punch
W	elastic potential per unit volume	a	radius of a punch in initially deformed body
e_{ii}	incremental strain	s_{ij}	incremental stress referred to axes which are incrementally displaced with the medium
λ_i	extension ratio	Δf_i	incremental boundary force per unit initial area
ϕ	displacement function	$C_1(\xi), C_2(\xi)$	integral constants

have been discussed by Inove et al. (1990), Sokamoto et al. (1990), Babich et al. (2004), Yang (2005) and Nadler and Tang (2008). In linear elastic fracture mechanics analysis, determination of the stresses is always a major consideration and has to be evaluated by using Hankel's transformation. Kuo and Keer (1992) used the Hankel transform to solve numerically the contact problem of a layered transversely isotropic half space. Sneddon, (1975) has given an application of integral transform techniques. Recently various powerful techniques such as decomposition method (Khan, 2009; Khan and Faraz, 2010, 2011a,b), homotopy perturbation method (Yildirim, 2010a,b) and variational methods (Faraz et al., 2011) have been proposed for obtaining exact and approximate solutions. Many results obtained in the literature regarding punch such as a sequential punch of flat-ended and wedge-shaped profile with crack initiating at one end of the contact region in Hasebe et al. (1989, 1990), who used a rational mapping function and complex stress function.

Fan and Chyanbin (1996) have solved punch problems by combining Stork's formalism and the method of analytic continuation. Vibration of an elliptic plate with variable thickness is discussed in Singh and Tyagi (1985) and Singh and Goel (1985). Recently, an analysis, based on finite element approach is given in Anifantis (2001) and initiation and propagation of surface cracks in Dag and Erdogan (2002). The present problem has been solved with the help of the theory of Bessel's function.

The present work is directed towards obtaining stresses for flat-ended circular cylindrical punch for initially stressed Neo-Hookean solids. The purpose of this paper is to obtain analytic expressions for stresses and displacements. These expressions of stresses and displacement containing infinite integrals and involving Bessel's function are solved numerically. After that the characteristics of such numerical modeling are discussed graphically. A semi-infinite initially stressed elastic medium is used, which is pressed normally by an axi-symmetric rigid punch. The medium is supposed to be isotropic, homogenous and incompressible.

2. Formulation of the problem

The equation of motion for incremental deformation theory in rectangular cartesian co-ordinates x_i and t is

$$\frac{\partial s_{ij}}{\partial x_j} + S_{jk}\frac{\partial w_{ik}}{\partial x_j} + S_{ik}\frac{\partial w_{jk}}{\partial x_j} - e_{jk}\frac{\partial S_{ik}}{\partial x_j} = \rho\frac{\partial^2 u_i}{\partial t^2} \tag{1}$$

The expression for incremental boundary force per unit area is

$$\Delta f_i = (s_{ij} + S_{jk}w_{ik} + S_{ij}e - S_{ik}e_{jk})n_j, \tag{2}$$

where the usual convention for summation over repeated indices is applied.

The elastic potential per unit volume for the material (so called Neo-Hookean solid) is expressed in the form given below according to Trefftz (1933)

$$W = \frac{1}{2}\mu_0(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3), \tag{3}$$

with

$$\lambda_1\lambda_2\lambda_3 = 1. \tag{4}$$

With the help of Eqs. (3) and (4), the stress–strain relations are

$$S_{11} - S_{22} = \mu_0(\lambda_1^2 - \lambda_2^2) \tag{5}$$

$$S_{22} - S_{33} = \mu_0(\lambda_2^2 - \lambda_3^2) \tag{6}$$

$$S_{33} - S_{11} = \mu_0(\lambda_3^2 - \lambda_1^2) \tag{7}$$

The equation of motion is reduced in cylindrical polar co-ordinates (r, θ, z) from rectangular cartesian co-ordinates (x_i, t) that are connected as

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \tan^{-1}(x_2/x_1), \quad z = x_3 \tag{8}$$

The components $S_{rr}, S_{\theta\theta}$ and S_{zz} of initial stress are assumed to be non-zero and are uniform throughout the body and the body is assumed in the state of symmetrical incremental strain with respect to the z-axis. Therefore, the Eq. (1) reduces in cylindrical polar co-ordinates to

$$\frac{\partial s_{rr}}{\partial r} + \frac{s_{rr} - s_{\theta\theta}}{r} + \frac{\partial s_{rz}}{\partial z} - (S_{rr} - S_{zz})\frac{\partial w_{rz}}{\partial z} = \rho\frac{\partial^2 u_r}{\partial t^2}, \tag{9}$$

$$\frac{\partial s_{zz}}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rs_{zr}) - (S_{rr} - S_{zz})\frac{1}{r}\frac{\partial}{\partial r}(rw_{rz}) = \rho\frac{\partial^2 u_z}{\partial t^2}. \tag{10}$$

The expressions for incremental displacement u_r and u_z in terms of scalar function $\Phi(r, z)$ are given by

$$u_r = -\frac{\partial^2 \Phi}{\partial r \partial z}, \tag{11}$$

$$u_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right). \tag{12}$$

The function Φ is given by the simple partial differential equation as follows