Journal of King Saud University – Science (2012) 24, 271–276

King Saud University
Journal of King Saud University – Science
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ORIGINAL ARTICLE

Optical quasi-solitons by Lie symmetry analysis

Anjan Biswas [a,*], Chaudry Masood Khalique [b]

[a] Department of Mathematical Sciences, Delaware State University, Dover, DE 19901-2277, USA
[b] International Institute for Symmetry Analysis and Mathematical Modeling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X2046, Mmabatho 2735, South Africa

Received 16 April 2011; accepted 12 May 2011
Available online 18 May 2011

KEYWORDS
Quasi-solitons;
Integrability;
Nonlinearity;
Kerr law;
Power law;
Parabolic law;
Saturable law;
Logarithmic law

Abstract This paper studies optical quasi-solitons by the aid of Lie group analysis. Nine types of nonlinearities are considered here. They are Kerr law, power law, parabolic law, dual-power law, polynomial law, triple-power law, saturable law, exponential law and log law nonlinearity. A closed form solution is obtained in each case.

1. Introduction

The nonlinear Schrödinger's equation (NLSE) plays a very important role in the area of Nonlinear Optics (Biswas and Konar, 2006; Biswas et al., 2008; Biswas and Milovic, 2010; Khalique and Biswas, 2009, 2010; Kohl et al., 2008, 2009; Kudryashov and Loguinova, 2009; Liu et al., 2010; Lü et al., 2008). This equation is the total backbone of the study of solitons that propagate through optical fibers for trans-oceanic and trans-continental distances. There are various kinds of nonlinearities that are studied in this context. In various kinds of optical fibers, there are these various kinds of optical nonlinearities that appear. The details of these types of nonlinearities are given in the book by Biswas and Konar that was published in 2006 (Biswas and Konar, 2006). In this paper, the focus will be on obtaining the solutions for nine types of nonlinear media. The perturbed NLSE that will be studied in this paper appears in the study of optical quasi-solitons.

There are various methods of studying the NLSE that has been developed in the past couple of decades. Some of these well known methods are Adomian decomposition method, He's variational iteration method, He's semi-inverse variational principle, exponential function method and many more. These methods have turned out to be a blessing in this area of research. However, one needs to be careful in applying these methods of integration as pointed out by Kudryashov in 2009 (Kudryashov and Loguinova, 2009). In this paper, the method of Lie symmetry, also known as Lie group analysis will be used to carry out the integration of the perturbed NLSE that governs the study of optical quasi-solitons (Kohl et al., 2008).

2. Mathematical analysis

The dimensionless form of the NLSE that is going to be studied in this paper is given by (Biswas and Konar, 2006; Kohl et al., 2008)

* Corresponding author.
E-mail address: biswas.anjan@gmail.com (A. Biswas).
Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2011.05.003

$$iq_t + aq_{xx} + bG(|q|^2)\,q = 0 \quad (1)$$

In (1), G is a real-valued algebraic function, where $G(|q|^2)q : C \mapsto C$. Considering the complex plane C as a two-dimensional linear space R^2, it can be said that the function $G(|q|^2)\,q$ is k times continuously differentiable so that one can write (Biswas and Konar, 2006; Kohl et al., 2009; Kudryashov and Loguinova, 2009)

$$G(|q|^2)\,q \in \bigcup_{m,n=1}^{\infty} C^k((-n,n)\times(-m,m);R^2)$$

In Eq. (1), q is the dependent variable while x and t are the independent variables that represent space and time respectively. The first term in (1) represents the time evolution term while the second term is due to the group velocity dispersion and the third term accounts for nonlinearity that is also known as the non-Kerr law of nonlinearity. This is a nonlinear partial differential equation that is not integrable, in general. The non-integrability is not necessarily related to the nonlinear term in (1). Higher order dispersion, for example, can also make the system non-integrable while it still remains Hamiltonian. The solutions to (1) for particular forms of G are known as *solitons*. These solitons are the outcome of a delicate balance between dispersion and nonlinearity.

The solution of (1) is given in the form (Biswas and Konar, 2006)

$$q(x,t) = Ag[B(x - vt)]e^{i(-\kappa x + \omega t + \theta)}, \quad (2)$$

where the function g represents the shape of the soliton that depends on the type of nonlinearity $G(s)$ in question. Here, A and B respectively represent the amplitude and width of the soliton while v is the soliton velocity, κ is the soliton frequency, ω is the soliton wave number and θ is the phase constant for the soliton. Thus,

$$\kappa = -v \quad (3)$$

and

$$\omega = \frac{i}{2E}\int_{-\infty}^{\infty}\left(qq_t^* - q^*q_t\right)\mathrm{d}x, \quad (4)$$

where in (4) E is the energy of the soliton that is defined in (5) in the following subsection and q^* denotes the complex conjugate of q.

2.1. Integrals of motion

An important property of NLSE given by (1) is that it has conserved quantities also known as *Integrals of Motion*. In fact, Eq. (1) has three integrals of motion. They are the energy (E), linear momentum (M) and Hamiltonian (H) which are respectively given by (Biswas and Konar, 2006; Kohl et al., 2008)

$$E = \int_{-\infty}^{\infty}|q|^2\,\mathrm{d}x, \quad (5)$$

$$M = i\int_{-\infty}^{\infty}(q^*q_x - qq_x^*)\,\mathrm{d}x \quad (6)$$

and

$$H = \int_{-\infty}^{\infty}\left[a|q_x|^2 - f(I)\right]\mathrm{d}x, \quad (7)$$

where

$$f(I) = \int_0^I F(\xi)\,\mathrm{d}\xi \quad (8)$$

with the intensity $I = |q|^2$. The first conserved quantity is also known as the *wave power* while mathematically, it is known as the L_2 norm. The Hamiltonian is one of the most fundamental notions in mechanics and more generally in the theory of conservative dynamical systems with finite or even infinite degrees of freedom. The most useful approach in the soliton theory of conservative non-integrable Hamiltonian system is a representation on the plane of conserved quantities namely the Hamiltonian-versus-energy diagrams.

3. Perturbation terms

The perturbed NLSE that is going to be studied in this paper is given by

$$iq_t + aq_{xx} + bG(|q|^2)q = kq_x^2q^*, \quad (9)$$

where k is the perturbation parameter. This Eq. (9) appears in the study of optical quasi-solitons (Kohl et al., 2008). In this paper, a different form of solution structure will be obtained. This solution form is given by (Khalique and Biswas, 2009, 2010)

$$q(x,t) = \phi(x)e^{i\lambda t} \quad (10)$$

On substituting the form (10), Eq. (9) reduces to the ordinary differential equation

$$\lambda\phi + a\phi'' + bG(\phi^2)\phi = k(\phi')^2\phi. \quad (11)$$

Eq. (11) has a single Lie point symmetry, namely $X = \partial/\partial x$ (Khalique and Biswas, 2009, 2010). This symmetry will be used to integrate equation (11) once. It can be easily seen that the two invariants are

$$u = \phi \quad (12)$$

and

$$v = \phi' \quad (13)$$

Treating u as the independent variable and v as the dependent variable, (11) can be rewritten as

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \left(\frac{ku}{a}\right)v - \left\{\frac{\lambda u + bG(u^2)u}{a}\right\}\frac{1}{v}. \quad (14)$$

Integrating (14) yields

$$v^2 = \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 = \frac{\lambda}{k} - \frac{2b}{a}e^{\frac{k\phi^2}{a}}\int^{\phi} sG(s^2)\,e^{-\frac{ks^2}{a}}\,\mathrm{d}s + c_1\,e^{\frac{k\phi^2}{a}}, \quad (15)$$

where c_1 is an arbitrary constant of integration. Now, (15) can be integrated once more to yield

$$x + c_2 = \int \frac{\mathrm{d}\phi}{\left[\frac{\lambda}{k} - \frac{2b}{a}e^{\frac{k\phi^2}{a}}\int^{\phi} sG(s^2)\,e^{-\frac{ks^2}{a}}\,\mathrm{d}s + c_1\,e^{\frac{k\phi^2}{a}}\right]^{\frac{1}{2}}}, \quad (16)$$

where c_2 is the second constant of integration. This equation will be further analyzed depending on the type of nonlinearity in the following subsections.