Journal of King Saud University – Science (2012) **24**, 289–294

King Saud University
Journal of King Saud University – Science
www.ksu.edu.sa
www.sciencedirect.com

ORIGINAL ARTICLE

On strict common fixed points of hybrid mappings in 2-metric spaces

V. Popa [a], M. Imdad [b], Javid Ali [c,*]

[a] *Department of Mathematics, University of Bacău, 600114 Bacău, Romania*
[b] *Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India*
[c] *Departament de Matemàtica Aplicada III, Universitat Politècnica de Catalunya, 08222 Terrassa, Barcelona, Spain*

Received 18 May 2011; accepted 21 July 2011
Available online 29 July 2011

KEYWORDS
2-Metric spaces;
Common fixed point;
OWC hybrid pair of mappings;
Implicit relation

Abstract In this paper, we introduce an implicit relation with a view to cover several contractive conditions in one go and utilize the same to prove a general common fixed point theorem for two hybrid pairs of occasionally weakly compatible mappings defined on 2-metric spaces. Our results extend, generalize and unify several known common fixed point theorems of the existing literature.
© 2011 King Saud University. Production and hosting by Elsevier B.V. All rights reserved.

1. Introduction

The concept of 2-metric spaces was introduced and investigated by Gähler in his papers (Gähler, 1963; Gähler, 1965) which were later developed by many other mathematicians including Gähler himself. Like various other aspects of the theory, a number of authors also studied a multitude of results of metric fixed point theory in the setting of 2-metric spaces. In doing so, the authors are indeed motivated by various concepts already known in respect of metric spaces which enable them to introduce analogous concepts in the frame work of 2-metric spaces. For this kind of work, we refer to Cho et al. (1988), Murthy et al. (1992), Tan et al. (2003), Naidu and Prasad (1986), Abu-Donia and Atia (2007), Pathak et al. (1995) wherein the weak conditions of commutativity such as: compatible mappings, compatible mappings of type (A) and type (P), weakly compatible mappings of type (A) and weakly compatible mappings were lifted to the setting of 2-metric spaces which were subsequently utilized to prove results on common fixed points in 2-metric spaces.

On the other hand, Al-Thagafi and Shahzad (2008) introduced the notion of occasional weak compatibility (in short OWC) as a generalization of weak compatibility. Jungck and Rhoades (2006) utilized this notion of OWC to prove common fixed point theorems in symmetric spaces. In fact, OWC is not a proper generalization of weak compatibility for those pairs of mappings whose set of coincidence points is empty. Imdad et al. (2011) pointed out that OWC is pertinent in respect of nontrivial weak compatible pairs (i.e., pairs with at least one coincidence point). In the same spirit, Pant and Pant (2010) redefined OWC and termed it as conditional commutativity wherein authors assumed that the set of coincidence points is nonempty. Most recently, Doric et al. (2011) proved that

* Corresponding author. Tel.: +34 698503111.
E-mail addresses: vpopa@ub.ro (V. Popa), mhimdad@yahoo.co.in (M. Imdad), javid@math.com (J. Ali).
Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

1018-3647 © 2011 King Saud University. Production and hosting by Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.jksus.2011.07.001

OWC and weak compatibility are identical notions in respect of single-valued pairs of mappings whenever point of coincidence is unique. But, the same is not true for pairs of hybrid mappings, i.e., OWC property is weaker than weak compatibility in respect of hybrid pairs of mappings.

2. Preliminaries

A 2-metric space is a set X equipped with a real-valued function d on X^3 which satisfies the following conditions:

(M_1) to each pair of distinct points x, y in X, there exists a point $z \in X$ such that $d(x,y,z) \neq 0$,
(M_2) $d(x,y,z) = 0$ when at least two of x,y,z are equal,
(M_3) $d(x,y,z) = d(x,z,y) = d(y,z,x)$,
(M_4) $d(x,y,z) \leqslant d(x,y,u) + d(x,u,z) + d(u,y,z)$ for all $x,y,z,u \in X$.

The function d is called a 2-metric on the set X whereas the pair (X,d) stands for 2-metric space. Geometrically, in respect of a 2-metric d, $d(x,y,z)$ represents the area of a triangle with vertices x, y and z.

It is known (cf. Gähler, 1965; Naidu and Prasad, 1986) that a 2-metric d is a non-negative continuous function in any one of its three arguments but the same need not be continuous in two arguments. A 2-metric d is said to be continuous if it is continuous in all of its arguments. Throughout this paper d stands for a continuous 2-metric.

Definition 2.1. A sequence $\{x_n\}$ in a 2-metric space (X,d) is said to be convergent to a point $x \in X$ (denoted by $\lim_{n\to\infty}x_n = x$) if $\lim_{n\to\infty}d(x_n,x,z) = 0$ for all $z \in X$.

Definition 2.2. A sequence $\{x_n\}$ in a 2-metric space (X,d) is said to be Cauchy sequence if $\lim_{n,m\to\infty}d(x_n,x_m,z) = 0$ for all $z \in X$.

Definition 2.3. A 2-metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Remark 2.1 (Naidu and Prasad, 1986). In general, a convergent sequence in a 2-metric space (X,d) need not be Cauchy, but every convergent sequence is a Cauchy sequence whenever 2-metric d is continuous on X.

Definition 2.4 (Murthy et al., 1992). A pair of self mappings (S,T) of a 2-metric space (X,d) is said to be compatible if $\lim_{n\to\infty} d(STx_n,TSx_n,z) = 0$ for all $z \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty}Tx_n = t$ for some $t \in X$.

Definition 2.5. A pair of self mappings (S,T) of a nonempty set X is said to be weakly compatible if $Sx = Tx$ (for some $x \in X$) implies $STx = TSx$.

Let (X,d) be a 2-metric space. We denote by $B(X)$, the family of bounded subsets of (X,d). For all A, B and C in $B(X)$, let $D(A,B,C)$ and $\delta(A,B,C)$ be the functions defined by

$$D(A,B,C) = \inf\{d(a,b,c) : a \in A, b \in B, c \in C\},$$
$$\delta(A,B,C) = \sup\{d(a,b,c) : a \in A, b \in B, c \in C\}.$$

If A consists of a single point 'a', we write $\delta(A,B,C) = \delta(a,B,C)$. Further, if B and C also consist of single points 'b' and 'c', respectively, then we write $\delta(A,B,C) = D(a,b,c) = d(a,b,c)$.

It follows from the definition that

$\delta(A,B,C) = 0$ if at least two A, B, C are identically equal and singleton,

$$\delta(A,B,C) = \delta(A,C,B) = \delta(B,A,C) = \delta(B,C,A) = \delta(C,B,A) = \delta(C,A,B) \geqslant 0,$$
$$\delta(A,B,C) \leqslant \delta(A,B,E) + \delta(A,E,C) + \delta(E,B,C) \text{ for all } A,B,C,E \text{ in } B(X).$$

Definition 2.6. A sequence $\{A_n\}$ of subsets of a 2-metric space (X,d) is said to be convergent to a subset A of X if:

(i) given $a \in A$, there exists $\{a_n\}$ in X such that $a_n \in A_n$ for $n = 1,2,3,\ldots$ and $\lim_{n\to\infty}d(a_n,a,z) = 0$ for each $z \in X$, and
(ii) given $\epsilon > 0$, there exists a positive integer N such that $A_n \subset A_\epsilon$ for $n > N$ where A_ϵ is the union of all open balls with centers in A and radius ϵ.

Definition 2.7. The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be weakly commuting at x if $IFx \in B(X)$ and

$$\delta(FIx, IFx, z) \leqslant \max\{\delta(Ix, Fx, z), \delta(IFx, IFx, z)\}. \tag{2.1}$$

Remark 2.2. If F is a single-valued mapping, then the set IFx becomes singleton. Therefore, $\delta(IFx, IFx, z) = 0$ and condition (2.1) reduces to the condition given by Khan (1984), that is $D(FIx, IFx, z) \leqslant D(Ix, Fx, z)$.

Definition 2.8. The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be compatible if $\lim_{n\to\infty}D(FIx_n, IFx_n, z) = 0$ for all $z \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}Ix_n = t \in A = \lim_{n\to\infty} Fx_n$ for some $t \in X$ and $A \in B(X)$.

Definition 2.9. The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be δ-compatible if $\lim_{n\to\infty}\delta(FIx_n, IFx_n, z) = 0$ for all $z \in X$, whenever $\{x_n\}$ is a sequence in X such that $IFx_n \in B(X)$, $Fx_n \to \{t\}$ and $Ix_n \to t$ for some $t \in X$.

Definition 2.10. Let $I : X \to X$ and $F : X \to B(X)$. A point $x \in X$ is said to be a fixed point (strict fixed point) of F if $x \in Fx$ ($Fx = \{x\}$). Also, a point $x \in X$ is said to be a coincidence point (strict coincidence point) of (I,F) if $Ix \in Fx$ ($Fx = \{Ix\}$).

Definition 2.11 (Jungck and Rhoades, 1998). The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be weakly compatible if they commute at all strict coincidence points, i.e., for each x in X such that $Fx = \{Ix\}$, we have $FIx = IFx$.

Remark 2.3 (Jungck and Rhoades, 1998). Any δ-compatible pair (I,F) is weakly compatible but not conversely.

Download English Version:

https://daneshyari.com/en/article/827497

Download Persian Version:

https://daneshyari.com/article/827497

Daneshyari.com