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Some resolvent methods for general variational inclusions

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Abstract In this paper, we consider and analyze some classes of resolvent-splitting methods for solving the general variational inclusions using the technique of updating the solution. These resolvent-splitting methods are self-adaptive-type methods, where the corrector step size involves the resolvent equation. We prove that the convergence of these new methods only require the pseudomonotonicity, which is a weaker condition than monotonicity. These new methods differ from the previously known splitting and inertial proximal methods for solving the general variational inclusions and related complementarity problems. The proposed methods include several new and known methods as special cases. Our results may be viewed as refinement and improvement of the previous known methods.

1. Introduction

Variational inequalities theory has played a significant and fundamental role in the development of new and innovative techniques for solving complex and complicated problems arising in pure and applied sciences, see (Alvarez, 2000; Alvarez and Attouch, 2001; Brezis, 1973; El Farouq, 2001; Giannessi and Maugeri, 1995; Giannessi et al., 2001; Glowinski and Tallec, 1989; Haugruge et al., 1998; He and Liao, 2002; Kinderlehrer and Stampacchia, 2000; Moudafi and Noor, 1999; Moudafi and Thera, 1997; Noor, 1988, 1993, 1997a,b, 1998, 2000, 2001a,b, 2002a,b, 2003, 2004, 2006a,b, 2009a,b,c, 2010a–d; Noor and Noor, 2010, 2004; Noor et al., 1993; Noor and Rassias, 2002; Patriksson, 1998; Shi, 1991; Stampacchia, 1964; Tseng, 2000; Uko, 1998; Xiu et al., 2001). Variational inequalities have been extended and generalized in various directions using novel and innovative techniques. A useful and important generalization is called the general variational inclusion involving the sum of two nonlinear operators T and A. Moudafi and Noor (1999) studied the sensitivity analysis of variational inclusions by using the technique of the resolvent equations. Recently much attention has been given to develop iterative algorithms for solving the variational inclusions. It is known that such algorithms require an evaluation of the resolvent operator of the type $(I + \rho(T + A))^{-1}$. The main difficulty with such problems is that the resolvent operator may be hard to

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invert. This difficulty has been overcome by using the resolvent operators $(I+\rho T)^{-1}$ and $(I+\rho A)^{-1}$ separately rather than $(I+\rho(T+A))^{-1}$. Such a technique is called the splitting method. These methods for solving variational inclusions have been studied extensively, see, for example (Glowinski and Tallec, 1989; Moudafi and Thera, 1997; Noor, 1998, 2000, 2001a,b, 2002a,b, 2003, 2004, 2006a,b, 2009a,b,c, 2010a) and the references therein. In the context of the mixed variational inequalities (variational inclusions), Noor (2000, 2001a, 2002b, 2003, 2004) has used the resolvent operator and resolvent equations techniques to suggest and analyze a number of resolvent type iterative methods. A useful feature of these splitting methods is that the resolvent step involves the subdifferential of the proper, convex and lower-semicontinuos function only and the other part facilitates the problem decomposition.

Noor (1998) introduced and considered the general variational inclusion, which is an important and significant generalization of variational inequalities. It turned out that a wide class of nonsymmetric, odd-order free, moving, unilateral and equilibrium problems arising in elasticity, transportation, circuit analysis, oceanography, nonlinear optimization, finance, economics and operations research can be studied via general variational inclusions, see (Alvarez, 2000; Alvarez and Attouch, 2001; Brezis, 1973; El Farouq, 2001; Giannessi and Maugeri, 1995; Giannessi et al., 2001; Glowinski and Tallec, 1989; Haugruge et al., 1998; He and Liao, 2002; Kinderlehrer and Stampacchia, 2000; Moudafi and Noor, 1999; Moudafi and Thera, 1997; Noor, 1988, 1993, 1997a,b, 1998, 2000, 2001a,b, 2002a,b, 2003, 2004, 2006a,b, 2009a,b,c, 2010a–d; Noor and Noor, 2010, 2004; Noor et al., 1993; Noor and Rassias, 2002; Patriksson, 1998; Shi, 1991; Stampacchia, 1964; Tseng, 2000; Uko, 1998; Xiu et al., 2001). Variational inclusion theory is experiencing an explosive growth in both theory and applications: as consequence, several numerical techniques including resolvent operator, resolvent equations, auxiliary principle, decomposition and descent are being developed for solving various classes of variational inclusions and related optimization problems. Resolvent methods and its variants forms including the resolvent equations represent important tools for finding the approximate solution of variational inclusions. The main idea in this technique is to establish the equivalence between the variational inclusions and the fixed-point problem by using the concept of resolvent operator. This alternative formulation has played a significant part in developing various resolvent methods for solving variational inclusions. It is well known that the convergence of the resolvent methods requires that the operator must be strongly monotone and Lipschitz continuous. Unfortunately these strict conditions rule out many applications of this method. This fact motivated to modify the resolvent method or to develop other methods. The extragradient method overcome this difficulty by performing an additional forward step and a projection at each iteration according to the double resolvent. This method can be viewed as predictor-corrector method. Its convergence requires that a solution exists and the monotone operator is Lipschitz continuous. When the operator is not Lipschitz continuous or when the Lipschitz continuous constant is not known, the extraresolvent method and its variant forms require an Armijo-like line search procedure to compute the step size with a new projection need for each trial, which leads to expansive computations. To overcomes these difficulties, several modified resolvent and extraresolvent-type methods have been suggested and developed for solving variational inequalities, see (Noor, 1998, 2000, 2001a,b, 2002a,b, 2003, 2004, 2006a,b, 2009a,b,c, 2010a) and the references therein. Glowinski and Tallec (1989) has suggested and analyzed some three-step splitting methods for solving variational inclusions problems by using the Lagrange multipliers technique. They have shown that three-step splitting are numerically more efficient as compared with one-step and two-step splitting methods. They have studied the convergence of these splitting methods under the assumption that the underlying operator is monotone and Lipschitz continuous. Noor (2004) has suggested some three-step projection-splitting methods for various classes of variational inequalities and variational inclusions using the technique of updating the solution, in which the order of T and $J_A + = (I+\rho A)^{-1}$, resolvent operator associated with the maximal monotone operator A, has not been changed. These three-step splitting methods are compatible with the three-step splitting methods of Glowinski and Tallec (1989). For the applications and convergence analysis of three-step splitting method, see (Glowinski and Tallec, 1989; He and Liao, 2002; Moudafi and Thera, 1997; Noor, 1998, 2000, 2001a,b, 2002a,b, 2003, 2004) and the references therein.

In this paper, we suggest and analyze a class of self-adaptive resolvent methods by modifying the fixed-point equations involving a generalized residue vector associated with the variational inclusions. These methods are simple and robust. The searching direction in these methods is a combination of the generalized resolvent residue and the modified extraresolvent direction. These new methods are different from the existing one-step, two-step and three-step projection-splitting methods. We prove that the convergence of the proposed methods only requires the pseudomonotonicity, which is weaker condition than monotonicity.

Noor (2004) and El Farouq (2001) has used the auxiliary principle technique to suggest and analyze a class of proximal (implicit) methods. Alvarez (2000) and Alvarez and Attouch (2001) have considered an inertial proximal method for maximal monotone operators via the discretization of a second order differential equation in time, which includes the classical proximal method. We again use the equivalent fixed-point formulation of the variational inclusions to suggest an inertial proximal method for general variational inclusions. We show that the convergence of the inertial proximal method requires the pseudomonotonicity, which is a weaker condition than monotonicity. Thus it is clear that our results improve the convergence criteria of the inertial proximal methods of Alvarez and Attouch (2001). Our proof of convergence is very simple as compared with other methods. Since general variational inclusions include classical variational inclusions and general (quasi) complementarity problems as special cases, results obtained in this paper continue to hold for these problems. The comparison of these methods with the existing ones is an interesting problem for future research work.

2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a closed convex set in H and $T, g : H \to H$ be nonlinear operators. Let $\varphi : H \to R \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function.