ORIGINAL ARTICLE

Recurrence relations for the moments of order statistics from doubly truncated modified Makeham distribution and its characterization

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Abstract In this study a general form of recurrence relations of continuous function for doubly truncated modified Makeham distribution is obtained. Recurrence relations between single and product moments of order statistics from doubly truncated modified Makeham distribution are given. Also, a characterization of modified Makeham distribution from the right and the left is discussed through the properties of order statistics.

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1. Introduction

Order statistics arise obviously in many real life applications. The moments of order statistics have assumed considerable interest in recent years and have been tabulated quite widely for several distributions. For an extensive survey, see for example (Arnold et al., 1992; Hendi et al., 2006). Many authors have presented some recurrence relations satisfied by single and product moments of order statistics see for example (Khan et al., 1983a,b; Athar and Islam, 2004; among others). A number of recurrence relations satisfied by these moments of order statistics are available in the literature. Balakrishnan and Malik (1985) derived some identities involving the density functions of order statistics. These identities are useful in checking the computation of the moments of order statistics. Balakrishnan and Malik (1986) established some recurrence relations of order statistics from the linear-exponential distribution. Balakrishnan et al. (1988) reviewed several recurrence relations and identities for the single and product moments of order statistics from some specific distributions. Recently, Bekçi (2009) considered some recurrence relations for the moments of order statistics from uniform distribution.

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Acronyms and Notations

MMD	modified Makeham distribution	b	MMD shape parameter ($b > 0$)
$h(x)$	hazard rate function	a	MMD scale parameter ($a > 0$)
$f(x)$	density function		
$F(x)$	cumulative distribution function		

Makeham distribution is an important life distribution and has been commonly used to fit actuarial data (see, Marshall and Olkin, 2007). According to Scollnik (1995), Makeham distribution of mortality represents a failure law, where the hazard rate is a mixture of non-aging failure distribution and the aging failure distribution with exponential increase in failure rates. For a description on the genesis and applications of Makeham distribution one may refer to Makeham (1860).

Feng et al. (2008) used the least squares type estimation to estimate the parameters of Makeham distribution. Aboutahoun and Al-Otaibi (2009) discussed relations between single and product moments of Makeham distribution.

The distribution presented in this paper is called modified Makeham distribution (MMD). This distribution has frequently been used to describe human mortality and to establish actuarial tables. Its hazard rate function is a product of a monotonically decreasing and a monotonically increasing function. The hazard rate function is given by

$$h(x) = \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} \exp\left[\left(\frac{x}{a}\right)^b\right], \tag{1.1}$$

where a is the scale parameter and b is the shape parameter. It is noted that for $b < 1$ the hazard rate function has a bathtub curve. This is a very desirable characteristic of the reliability models. It is important to note that the scale parameter a can be taken equal to 1 with no loss in generality. This simplifies many of the expressions.

Hazard rate function is the conditional probability that failure of the device will occur during a small interval $[t; t + \Delta t]$ given that device has survived to time t. The bathtub hazard rate function is typical for the lifetime description of most technical devices. At the beginning the failure rate decreases, after a while it becomes stable, and then it increases due to natural wear out processes.

The cumulative distribution function of the MMD is of the following form:

$$F(x) = 1 - \exp\left[1 - \exp\left(\frac{x}{a}\right)^b\right]. \tag{1.2}$$

Derivative of the cumulative distribution function (1.2) gives density function of the following form:

$$f_X(x) = \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} \exp\left[\left(\frac{x}{a}\right)^b\right] \exp\left[1 - \exp\left(\frac{x}{a}\right)^b\right]. \tag{1.3}$$

For basic characteristics of this distribution see (Kosznik-Biernacka, 2006, 2007, 2011).

In this paper, some recurrence relations satisfied by single and product moments of order statistics from doubly truncated MMD are established. Recurrence relations for the non-truncated case are given as a special case. A characteriza-

tion of this distribution based on the properties of the order statistics has also been considered.

The doubly truncated probability density function of the MMD is given by

$$f(x) = \frac{1}{P - Q}\left(\frac{b}{a}\right)\left(\frac{x}{a}\right)^{b-1} \exp\left[\left(\frac{x}{a}\right)^b\right] \exp\left[1 - \exp\left(\frac{x}{a}\right)^b\right],$$
$$Q_1 \leqslant x \leqslant P_1. \tag{1.4}$$

where $\quad 1 - P = \exp\left[1 - \exp\left\{\left(\frac{P_1}{a}\right)^b\right\}\right] \quad$ and $\quad 1 - Q = \exp\left[1 - \exp\left\{\left(\frac{Q_1}{a}\right)^b\right\}\right]$

Let $P_2 = \frac{1-P}{P-Q}$ and $Q_2 = \frac{1-Q}{P-Q}$.

From (1.4), we have

$$1 - F(x) = -P_2 + \left(\frac{a}{b}\right)\left(\frac{a}{x}\right)^{b-1} e^{-\left(\frac{x}{a}\right)^b} f(x). \tag{1.5}$$

Let $X_{1:n} \leqslant X_{2:n} \leqslant \cdots \leqslant X_{n:n}$ be the order statistics from continuous distribution functions (c.d.f) $F_1(x)$ and probability density function (p.d.f) $f_1(x)$. The p.d.f of order statistics $X_{r:n}$ and the joint p.d.f of two order statistics $X_{r:n}$ and $X_{s:n}$ are

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}[F_1(x)]^{r-1}[1 - F_1(x)]^{n-r}f_1(x),$$

$$f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$
$$[F_1(x)]^{r-1}[F_1(y) - F_1(x)]^{s-r-1}[1 - F_1(y)]^{n-s}f_1(x)f_1(y),$$
$$x < y; 1 \leqslant r < s \leqslant n.$$

Then for any monotone continuous function $\phi(x)$, we have

$$\alpha_{r:n} = E[\phi(x)]$$
$$= \frac{n!}{(r-1)!(n-r)!}$$
$$\times \int_{Q_1}^{P_1} \phi(x)[F(x)]^{r-1}[1 - F(x)]^{n-r}f(x)dx \tag{1.6}$$

and for any measurable joint function $\phi(x,y)$, we have

$$\alpha_{r,s:n} = E[\phi(x,y)] = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$
$$\times \int_{Q_1}^{P_1} \int_x^{P_1} \phi(x,y)[F(x)]^{r-1}[F(y) - F(x)]^{s-r-1}$$
$$\times [1 - F(y)]^{n-s}f(x)f(y)dydx. \tag{1.7}$$

Also, assume that