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ORIGINAL ARTICLE

Numerical computation of BCOPs[1] in two variables for solving the vibration problem of a CF-elliptical plate

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Abstract Boundary characteristic orthogonal polynomials in *xy*-coordinates have been built up over an elliptical domain occupied by a thin elastic plate. Half of the plate boundary is taken clamped while the other half is kept free. Coefficients of these polynomials have been computed once and for all so that an orthogonal polynomial sequence is generated from a set of linearly independent functions satisfying the essential boundary conditions of the problem. Use of this sequence in Rayleigh–Ritz method for solving the free vibration problem of the plate makes it faster in convergence and leads to a simplified system whose solution is comparatively easier. Three-dimensional solution surfaces and the associated contour lines have been plotted in some selected cases. Comparison have been made with known results whenever available.

1. Introduction

Use of orthogonal polynomials in the Rayleigh–Ritz method for solving most of the important differential equations has attracted the researcher's interest since 19th century. Many studies on the vibration of non-rectangular plates assuming various deflection shape functions in the Rayleigh–Ritz method have been reported by Leissa (1969). Following the publications of Szego's well known treatise Szego (1967) and Singh and Chakraverty (1991, 1992, 1993, 1994a,b) there has been tremendous growth of literature covering various aspects of the subject but, unfortunately, for plates of uniform boundary conditions. Sato (1973) presented experimental as well as theoretical results for elliptic plates but again with uniform free edge. An interesting contribution in this regard has been done by Bhat et al. (1998) and Chakraverty et al. (1999). They presented a recurrence scheme that makes the generation of two-dimensional boundary characteristic orthogonal polynomials for a variety of geometries straight forward and quite efficient. They also provide a survey of the application of BCOPs method in vibration problems. Some important books on orthogonal polynomials and its applications are Beckmann (1973), Chihara (1978), and Gautschi et al. (1999).

There is no analytical solution to the vibration problems of plates with non-uniform boundary conditions even for plates of simple geometrical shapes like rectangles (Wei et al., 2001).

[1] Boundary characteristic orthogonal polynomials.

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Notation

BCOPs	boundary characteristic orthogonal polynomials	ν	Poisson ratio
CC	for a plate with uniform fully clamped boundary	λ	non-dimensional frequency parameter
CF	for a plate with half of the boundary, $y \leqslant 0$, clamped and the rest free	∇^2	Laplacian operator
FF	for a plate with uniform completely free boundary	N	the approximation order
$a,\ b$	semi major and minor axes of the elliptical domain	c_j	the unknown coefficients used in the solution expansion
r	aspect ratio b/a	$\phi_i(X,Y)$	orthogonal functions over R
$x,\ y$	cartesian coordinates	$\hat{\phi}_i(X,Y)$	orthonormal functions over R
$X,\ Y$	non-dimensional coordinates $X = x/a,\ \ Y = y/a$	β_{ij}	coefficients of the orthogonal polynomials $\phi_i(X,Y)$
R'	domain occupied by the plate in xy-coordinate	$f,\ g$	functions of x and y
R	domain occupied by the plate in XY-coordinates	$\langle f,g\rangle$	inner product of f and g
$W(X,Y)$	displacement	$\Vert f\Vert$	norm of f
ρ	density of the material of the plate	$[a_{ij}],\ [b_{ij}]$	$N \times N$ matrices
E	Young's modulus		
ω	angular natural frequency		

Very little is available in literature on elliptical plates with non-uniform boundary conditions and, whenever available, it is mostly on circular plates. That is why this kind of problems has become a challenging problem for scientists and engineers. Some available references are Eastep and Hemmig (1982), Hemmig (1975), Leissa et al. (1979), Laura and Ficcadenti (1981) and Narita and Leissa (1981). Hassan (2007) has generated BCOPs to compute natural frequencies of an elliptical plate with half of the boundary simply supported and the rest free and gave numerical and graphical results for this case. In Hassan (2004) he solved the vibration problem under consideration by using traditional basis functions that satisfy the essential boundary conditions of the CF-elliptical plate in the Rayleigh–Ritz method. Explicit numerical and graphical results have been given and reported for the first time. Other publications dealing with plates with mixed boundary conditions have been recently appeared by Boborykin (2006), Czernous (2006), and Zovatto and Nicolini (2006). They investigated the bending problem of a rectangular plate with mixed boundary conditions. No numerical results are available for vibrations of elliptical plates with mixed boundary conditions.

The aim of the present work is to generate a sequence of boundary characteristic orthogonal polynomials over an elliptical domain occupied by a thin elastic plate with half of the boundary, $y \leqslant 0$, clamped and the rest free. These polynomials should satisfy at least the essential boundary conditions of the problem. The coefficients of these polynomials will be generated and tabulated in advance, once and for all, with the desired precision. Use of these polynomials in Rayleigh–Ritz method helped in presenting explicit numerical results for the problem under consideration. This method reduces ill-conditioning of the resulting system whose solution has become comparatively easier and faster in convergence. Three-dimensional solution surfaces, mode shapes, and the associated contour lines of the problem have been plotted in some selected cases. Comparison of results have been made with known results in literature whenever available.

2. Generation of boundary characteristic orthogonal polynomials

As has been done by Bhat (1985) for one-dimensional orthogonal polynomials and by Liew et al. (1990) for rectangular plates one will follow the same procedures to generate a set of orthogonal polynomials in two variables over an elliptical domain R' occupied by a thin elastic plate in the xy-plane with half of the boundary, $y \leqslant 0$, clamped and the rest free. For this one can start with the set of linearly independent functions

$$\{F_i(x,y) = uf_i(x,y)\}_{i=1}^{N}, \tag{1}$$

with (x,y) is a point in $R' = \left\{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1\right\}$ and $a,\ b$ as the semi major and semi minor axes of the elliptical domain. The functions u and f are chosen to be of the form

$$u = \begin{cases} (y^2 - r^2 z^2)^2 & \text{for CC-elliptical plate,} \\ (y + rz)^2 z & \text{for CF-elliptical plate,} \\ 1 & \text{for FF-elliptical plate,} \end{cases} \tag{2}$$

with $z = \sqrt{1 - x^2},\ \ r = \frac{b}{a}$, and

$$\{f_i, i = 1,2,\ldots\} = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, \ldots\}, \tag{3}$$

so that the essential boundary conditions of the problem are satisfied. To obtain an orthogonal set we define the inner product of two functions f and g by

$$\langle f,g\rangle = \int\int_{R'} f(x,y)g(x,y)\,dx\,dy \tag{4}$$

and the norm of a function f is then defined by

$$\Vert f\Vert^2 = \langle f,f\rangle = \int\int_{R'} f^2(x,y)\,dx\,dy \tag{5}$$

The orthogonal functions $\phi_i(x,y)$ are generated by using Gram–Schmidt process the algorithm for which may be summarized as follows:

$$\left.\begin{aligned} \phi_1 &= F_1, \\ \phi_i &= F_i - \sum_{j=1}^{i-1} \alpha_{ij}\phi_j, \\ \text{where} \quad & \\ \alpha_{ij} &= \langle F_i, \phi_j\rangle / \langle \phi_j, \phi_j\rangle, \quad j = 1,2,\ldots,i-1 \end{aligned}\right\}, \quad i = 2,3,\ldots,N. \tag{6}$$

The functions ϕ_i can be normalized by using the equation

$$\hat{\phi}_i = \phi_i / \Vert\phi_i\Vert = \phi_i / \langle \phi_i, \phi_i\rangle^{\frac{1}{2}}. \tag{7}$$