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The modified couple stress functionally graded Timoshenko beam formulation

M. Asghari *, M. Rahaeifard, M.H. Kahrobaiyan, M.T. Ahmadian

School of Mechanical Engineering, Sharif University of Technology, Tehran, Iran

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ABSTRACT

In this paper, a size-dependent formulation is presented for Timoshenko beams made of a functionally graded material (FGM). The formulation is developed on the basis of the modified couple stress theory. The modified couple stress theory is a non-classic continuum theory capable to capture the small-scale size effects in the mechanical behavior of structures. The beam properties are assumed to vary through the thickness of the beam. The governing differential equations of motion are derived for the proposed modified couple-stress FG Timoshenko beam. The generally valid closed-form analytic expressions are obtained for the static response parameters. As case studies, the static and free vibration of the new model are respectively investigated for FG cantilever and FG simply supported beams in which properties are varying according to a power law. The results indicate that modeling beams on the basis of the couple stress theory causes more stiffness than modeling based on the classical continuum theory, such that for beams with small thickness, a significant difference between the results of these two theories is observed.

1. Introduction

Functionally graded materials (FGMs) are produced from mixing of two different materials. This type of materials provides the specific benefits of both of the constituents. They can be defined as inhomogeneous composites which are made from a mixture of two different materials, usually a metal and a ceramic, with a desired continuous variation of properties as a function of position along certain dimension(s). The continuously compositional variation of the constituents in FGMs along different directions is the great benefit of FGMs, because this property offers a solution to the problem of appearing high magnitude shear stresses that may be induced in laminated composites, where two materials with great differences in properties are bonded. Nowadays, structures made of FGMs have a great practical role in engineering and industrial fields.

Some works have been performed by researchers on the static and dynamic behavior of beams and plates made of FGMs. Asghari et al. [1] have mentioned some instances of these works, including Refs. [2–7]. As another instance, the thermal snapping of functionally graded plates has been investigated by Prakash et al. [8]. Also, Jomehzadeh et al. [9] presented an analytical approach for the stress analysis of functionally graded annular sector plates. Moreover, analytical modeling of thermal residual stresses in some functionally graded material systems has been presented by Bouchafa et al. [10]. It is noted that these sample works are based on the classical continuum theory, while the formulation presented in this work is based on a non-classical continuum theory, the modified couple stress theory, which is discussed in detail in the following.

In recent years, the application of FG materials has broadly been spread in micro and nano structures such as thin films in the form of shape memory alloys [11,12], micro- and nano-electromechanical systems (MEMS and NEMS) [13,14] and also atomic force microscopes (AFMs) [15]. Beams used in MEMS, NEMS and AFMs, have the thickness in the order of microns and sub-microns, so that the small scale effects in their behavior is considerable. The size-dependent static and vibration behavior in micro scales are experimentally validated (see for example [16–19]). Considering experimental observations, it is well-known that size-dependent behavior is an inherent property of materials which appears for a beam when the characteristic size such as thickness or diameter is close to the internal material length scale parameter [20].

The classical continuum mechanics theories are not capable of prediction and explanation of the size-dependent behaviors which occur in micron- and sub-micron-scale structures. However, non-classical continuum theories such as higher-order gradient theories and the couple stress theory are acceptably able to interpret the size-dependencies.

In 1960s some researchers introduced the couple stress elasticity theory [21–23]. In the constitutive equation of this theory, some higher-order material length scale parameters appear in addition to the two classical Lame constants. Yang et al. [24] argued that in addition to the classical equilibrium equations of forces and moments of forces, another equilibrium equation should be considered for the material elements. This additional equation is

* Corresponding author. Tel.: +98 21 66165523; fax: +98 21 66000021.
E-mail address: asghari@sharif.edu (M. Asghari).

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the equilibrium of moments of couples. Then, they concluded that this additional equilibrium equation implies the symmetry of the couple stress tensor. Accordingly, they modified the constitutive equations of the couple stress theory and present the new constitutive equations. Utilizing the modified couple stress theory, Park and Gao [25] analyzed the static mechanical properties of an Euler–Bernoulli beam. Recently, Kong et al. [20] derived the governing equation, initial and boundary conditions of an Euler–Bernoulli beam based on the modified coupled stress theory using the Hamilton principle. Also, Asghari et al. [1] investigated the size-dependent behavior of FGM micro beams using the modified couple stress theory and the Euler–Bernoulli beam model.

The Timoshenko beam is a model for the study of behaviors of beams with less restrictive assumptions with respect to the Euler–Bernoulli beam. The normality assumption for sections in the Euler–Bernoulli beam model is discarded in the Timoshenko beam model. Hence, the Timoshenko beam is capable to capture the shear deformation in contrast to the Euler–Bernoulli beam. Although the Timoshenko beam is a complicated model with respect to the Euler–Bernoulli model, it possesses more capabilities and studying the behavior of beams based on the Timoshenko model gives closer results to the exact behavior. Recently in an interesting work, the modified couple stress theory is utilized by Ma et al. [26] in order to investigate the size-dependent behavior of a homogeneous Timoshenko beam. This work is indeed the generalization of the work of Ma et al. [26] to the FGM beams.

In this paper, considering both of bending and axial deformations, an FGM Timoshenko beam is proposed on the basis of the modified couple stress theory. In addition, generally valid closed-form analytic expressions are derived for the bending and axial deformations and also the angle of rotation of the cross sections in the static behavior. As a case study, response of a specific FGM cantilever beam subjected to a concentrated force at its free end is obtained. Further more, the natural frequency of a simply supported FGM beam is obtained and investigated in order to delineate the size-dependent vibration behavior of FGM Timoshenko beams.

2. Preliminaries

In the modified couple stress theory, the strain energy density for a linear elastic material in infinitesimal deformation is written as [24]

$$\bar{u} = \frac{1}{2}(\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij}) \quad (i,j = 1,2,3), \tag{1}$$

where for isotropic cases it is written

$$\sigma_{ij} = \lambda tr(\boldsymbol{\varepsilon})\delta_{ij} + 2\mu\varepsilon_{ij}, \tag{2}$$

$$\varepsilon_{ij} = \frac{1}{2}\left((\nabla\mathbf{u})_i + (\nabla\mathbf{u})_i^T\right), \tag{3}$$

$$m_{ij} = \beta\chi_{ij} = 2l^2\mu\chi_{ij}, \tag{4}$$

$$\chi_{ij} = \frac{1}{2}\left((\nabla\boldsymbol{\theta})_{ij} + (\nabla\boldsymbol{\theta})_{ij}^T\right), \tag{5}$$

in which σ_{ij}, ε_{ij}, m_{ij} and χ_{ij} denote the components of the symmetric part of stress tensor $\boldsymbol{\sigma}$, the strain tensor $\boldsymbol{\varepsilon}$, the deviatoric part of the couple stress tensor $\mathbf{m}$, and the symmetric part of the curvature tensor $\boldsymbol{\chi}$, respectively. Also, $\mathbf{u}$ and $\boldsymbol{\theta}$ are the displacement vector and the rotation vector noting that $\boldsymbol{\theta} = curl(\mathbf{u})/2$. The Lame constants and the material length scale parameter are represented by λ, μ and l, respectively. The parameter β is indeed a higher-order modulus which can be regarded as the rotational modulus which represents the resistance of the material against the gradient of the rotation of its elements. This parameter is related to the shear modulus μ and the length scale parameter l through $\beta = 2l^2\mu$. In order to determine parameter l for a specific material, some typical experiments such as micro-bend test, micro-torsion test and specially micro/nano indentation test can be carried out (see [16,17,19,27–29]).

The coordinate system, the kinematic parameters and the loading of a Timoshenko FG beam along the x-axis modeled on the basis of the modified couple stress theory are illustrated in Fig. 1. It is assumed the properties of the sections of the beam are not under variation along the axial coordinate x. For a Timoshenko beam, the displacement field is assumed as follows [30]

$$u_x = u(x,t) + z\psi(x,t), \quad u_y = 0, \quad u_z = w(x,t), \tag{6}$$

where u_x, u_y and u_z represent the displacement along x, y and z axes, respectively. Indeed, it is assumed that all cross sections remain plane after deformation; however, they can undergo a rigid body displacement in x–z plane and also a rotation about y-axis. Function $\psi(x,t)$ stands for the rotation angle of the beam cross-sections about y-axis. Also, function $u(x,t)$ denotes the axial displacement of points of a specific line in the section parallel to y-axis. This specific line, calling it the bending line, is the one when imposing a pure bending (without axial force resultant) on the section, no normal stress appears on it. Indeed, z is the distance of points of a section from its bending line. In Fig. 1, parameter f denotes the effect of axial body force imposed on the section as force per unit axial length. In this work it is assumed that the axial body forces in all sections have no moment resultant about the bending line. Also, parameter ω stands for the resultant of the transverse tractions on the top and bottom of the beam and also transverse body forces as force per unit axial length. Parameter c represents the resultant of y-component of the body couples imposed on the sections as couple per unit axial length. The distance of an arbitrary point from the bottom surface is represented by $\tilde{z}$. The distance between the bending line and the bottom surface is shown by $\tilde{z}_c$. Using Eqs. (3) and (6), the non-zero components of the strain tensor can be obtained as [26]

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + z\frac{\partial\psi}{\partial x}, \quad \varepsilon_{xz} = \frac{1}{2}\left(\psi + \frac{\partial w}{\partial x}\right). \tag{7}$$

Also, from $\boldsymbol{\theta} = curl(\mathbf{u})/2$, it can be written [26]

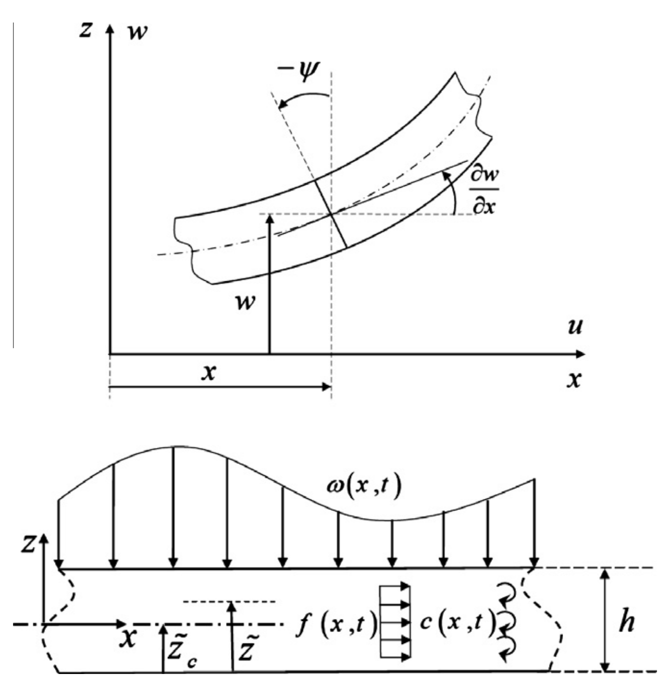

Fig. 1. Configuration, loading and coordinate system.