Materials and Design 29 (2008) 1518–1525

Contents lists available at ScienceDirect

Materials and Design

journal homepage: www.elsevier.com/locate/matdes

Three-dimensional free vibration analysis of functionally graded annular plates using the Chebyshev–Ritz method

C.Y. Dong *

Department of Mechanics, School of Science, Beijing Institute of Technology, Beijing 100081, China

ARTICLE INFO

Article history:
Received 14 December 2007
Accepted 3 March 2008
Available online 8 March 2008

Keywords:
Functionally graded annular plates
Chebyshev–Ritz method
Free vibration

ABSTRACT

The aim of this paper is to investigate three-dimensional free vibration of functionally graded annular plates with different boundary conditions using the Chebyshev–Ritz method, in which a set of duplicate Chebyshev polynomial series multiplied by the boundary function satisfying the boundary conditions are chosen as the trial functions of the displacement components. Two kinds of variations of material properties in the thickness direction of the plates are considered. Convergence of the Chebyshev–Ritz method is checked. Numerical results are given and compared with the previously published solutions.

1. Introduction

Functionally graded materials (FGMs) were first proposed in 1984 by Japanese material scientists [1]. FGMs possess properties that vary continuously as a function of position within the material, thus FGMs can be used to avoid interfacial stress concentrations appeared in laminated structures. Various applications of FGMs can be found in Refs. [2,3].

Compared with the analysis of functionally graded plates [4–8] and functionally graded spheres [9,10] as well as functionally graded cylindrical shells [11,12], the study of functionally graded circular and annular plates is very limited in number. Eraslan and Akis [13] obtained the closed-form solution of functionally graded rotating solid shaft and rotating solid disk under generalized plane strain and plane stress assumptions, respectively. Prakash and Ganapathi [14] analyzed the asymmetric flexural vibration and thermoelastic stability of FGMs circular plates using finite element method. Efraim and Eisenberger [15] studied the vibration of variable thickness annular isotropic plates and functionally graded plates. Nie and Zhong [16] investigated three-dimensional vibration of functionally graded circular plates using semi-analytical method.

The Chebyshev–Ritz method has successfully been used to carry out the analysis of free vibration of isotropic plates by Zhou et al. [17–20]. The advantage of the Chebyshev–Ritz method in solving free vibration of isotropic plates has been shown by Zhou et al. [17–20]. It can be expected that the Chebyshev–Ritz method can also be used to study free vibration of functionally graded plates. To the best of author's knowledge, the Chebyshev–Ritz method has not been yet adopted to study free vibration of functionally graded annular plates. Therefore in this paper, the formulation using the Chebyshev–Ritz method can be considered as an extension of isotropic plate vibration analysis and it is similar to Zhou et al. [17–20] but extended to three-dimensional free vibrations of functionally graded annular plates with different boundary conditions. Convergence of the Chebyshev–Ritz method is checked. The obtained results are compared with the previously published results.

2. Basic formulation

One functionally graded annular plate with inner radius R_0 and outer radius R_1 and thickness h is studied in this paper. A cylindrical coordinate system (r,θ,z) with the origin o at the center of the annular plate is used to describe the annular plate displacements, i.e. the radial direction displacement u, the circumferential direction displacement v and the thickness direction displacement w.

The linear elastic strain energy V and kinetic energy T for a functionally graded annular plate are given as follows:

$$V = \int_0^{2\pi} \int_{R_0}^{R_1} \int_{-h/2}^{h/2} \frac{1}{2} \boldsymbol{S}^{\mathrm{T}} \boldsymbol{C} \boldsymbol{S} r \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \tag{1}$$

* Tel./fax: +86 10 68912731/2735.
E-mail address: chunyingdong@yahoo.com

0261-3069/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.matdes.2008.03.001

and

$$T=\frac{1}{2}\int_0^{2\pi}\int_{R_0}^{R_1}\int_{-h/2}^{h/2}\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right]r\rho(z)\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \tag{2}$$

where

$$C=\frac{E(z)(1-v)}{(1+v)(1-2v)}\begin{bmatrix}1 & \frac{v}{1-v} & \frac{v}{1-v} & 0 & 0 & 0\\ \frac{v}{1-v} & 1 & \frac{v}{1-v} & 0 & 0 & 0\\ \frac{v}{1-v} & \frac{v}{1-v} & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2v}{2(1-v)} & 0 & 0\\ q0 & 0 & 0 & 0 & \frac{1-2v}{2(1-v)} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2(1-v)}\end{bmatrix} \tag{3}$$

in which for functionally graded materials with two constituent materials Poisson ratio v is assumed to be constant through the thickness, whereas the variations through the thickness of Young's modulus $E(z)$ and the mass density per unit volume $\rho(z)$ can be written as

$$E(z)=(E_\mathrm{m}-E_\mathrm{c})V_\mathrm{f}(z)+E_\mathrm{c} \tag{4a}$$

and

$$\rho(z)=(\rho_\mathrm{m}-\rho_\mathrm{c})V_\mathrm{f}(z)+\rho_\mathrm{c} \tag{4b}$$

where E_m and E_c denote the Young's moduli of the top and bottom materials, respectively; ρ_m and ρ_c denote the mass density per unit volume of the top and bottom materials, respectively; V_f is the volume fraction of the top material, and can be assumed to be the following form [11]:

$$V_\mathrm{f}(z)=\left(\frac{z}{h}+\frac{1}{2}\right)^g \tag{5}$$

in which z is the thickness coordinate ($-h/2\leqslant z\leqslant h/2$), and $g\geqslant 0$ is the gradient index.

Besides the above assumption of material properties, some researchers [16,21,22] assumed the exponential variation of material properties in the thickness direction of the plates, i.e.

$$E(z)=E_\mathrm{c}\mathrm{e}^{g\left(\frac{z}{h}+\frac{1}{2}\right)} \tag{6a}$$

and

$$\rho(z)=\rho_\mathrm{c}\mathrm{e}^{g\left(\frac{z}{h}+\frac{1}{2}\right)} \tag{6b}$$

where E_c, ρ_c, g, z and h are the same as those from Eqs. (4) and (5).

In Eq. (1), the matrix $\boldsymbol{S}$ is as follows:

$$\boldsymbol{S}=[S_{rr}\quad S_{\theta\theta}\quad S_{zz}\quad S_{\theta z}\quad S_{rz}\quad S_{r\theta}]^\mathrm{T} \tag{7}$$

where

$$\begin{aligned}&S_{rr}=\frac{\partial u}{\partial r},\quad S_{\theta\theta}=\frac{1}{r}\left(\frac{\partial v}{\partial\theta}+u\right),\quad S_{zz}=\frac{\partial w}{\partial z},\\&S_{r\theta}=\frac{1}{r}\frac{\partial u}{\partial\theta}+\frac{\partial v}{\partial r}-\frac{v}{r},\quad S_{rz}=\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z},\\&S_{\theta z}=\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial\theta}\end{aligned} \tag{8}$$

In numerical implementation, some dimensionless parameters are introduced [18]

$$\bar{r}=\frac{2r}{\overline{R}}-\delta,\quad \bar{\theta}=\theta,\quad \bar{z}=\frac{2z}{h} \tag{9}$$

where $\overline{R}=R_1-R_0$, $\delta=\frac{R_1+R_0}{R_1-R_0}$.

In the situation of free vibration, the displacement components of the annular plate can be expressed into the following forms:

$$\begin{aligned}&u(r,\theta,z,t)=U(\bar{r},\bar{\theta},\bar{z})\mathrm{e}^{\mathrm{i}\omega t},\quad v(r,\theta,z,t)=\widehat{V}(\bar{r},\bar{\theta},\bar{z})\mathrm{e}^{\mathrm{i}\omega t},\\&w(r,\theta,z,t)=W(\bar{r},\bar{\theta},\bar{z})\mathrm{e}^{\mathrm{i}\omega t}\end{aligned} \tag{10}$$

where ω is the eigenfrequency of the circular annular plate and $\mathrm{i}=\sqrt{-1}$.

Considering the circumferential symmetry of the circular annular plate about the coordinate θ, the displacement amplitude functions can be expressed as trigonometric functions in the circumferential direction as

$$\begin{aligned}&U(\bar{r},\bar{\theta},\bar{z})=\overline{U}(\bar{r},\bar{z})\cos(s\bar{\theta})\\&\widehat{V}(\bar{r},\bar{\theta},\bar{z})=\overline{V}(\bar{r},\bar{z})\sin(s\bar{\theta})\\&W(\bar{r},\bar{\theta},\bar{z})=\overline{W}(\bar{r},\bar{z})\cos(s\bar{\theta})\end{aligned} \tag{11}$$

where $s=0,1,\ldots,\infty$. As mentioned by Zhou et al. [18], $s=0$ means the axisymmetric vibration, i.e. $U(\bar{r},\bar{\theta},\bar{z})=U(\bar{r},\bar{z})$, $\widehat{V}(\bar{r},\bar{\theta},\bar{z})=0$, $W(\bar{r},\bar{\theta},\bar{z})=W(\bar{r},\bar{z})$. Rotating the symmetry axes by $\pi/2$, another set of free vibration modes can be obtained, corresponding to an interchange of $\cos(s\bar{\theta})$ and $\sin(s\bar{\theta})$ in Eq. (11). For this case, $s=0$ means the torsional vibration, i.e. $U(\bar{r},\bar{\theta},\bar{z})=0$, $\widehat{V}(\bar{r},\bar{\theta},\bar{z})=\overline{V}(\bar{r},\bar{z})$, $W(\bar{r},\bar{\theta},\bar{z})=0$.

Based on Eqs. (9)–(11), Eqs. (1) and (2) can be changed into the following forms:

$$\begin{aligned}V=\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}\Bigg\{&\frac{1}{2}[\overline{S}_{rr}\quad\overline{S}_{\theta\theta}\quad\overline{S}_{zz}\quad\overline{S}_{rz}]\begin{bmatrix}C_{11}&C_{12}&C_{12}&0\\C_{12}&C_{11}&C_{12}&0\\C_{12}&C_{12}&C_{11}&0\\0&0&0&C_{44}\end{bmatrix}\begin{Bmatrix}\overline{S}_{rr}\\\overline{S}_{\theta\theta}\\\overline{S}_{zz}\\\overline{S}_{rz}\end{Bmatrix}\Gamma_1\\&+\frac{1}{2}[\overline{S}_{\theta z}\quad\overline{S}_{r\theta}]\begin{bmatrix}C_{44}&0\\0&C_{44}\end{bmatrix}\begin{Bmatrix}\overline{S}_{\theta z}\\\overline{S}_{r\theta}\end{Bmatrix}\Gamma_2\Bigg\}(\bar{r}+\delta)\mathrm{d}\bar{z}\,\mathrm{d}\bar{r}\end{aligned} \tag{12}$$

and

$$T=\frac{\overline{R}^2h}{16}\omega^2\int_{-1}^{1}\int_{-1}^{1}(\Gamma_1\overline{U}^2+\Gamma_2\overline{V}^2+\Gamma_3\overline{W}^2)(\bar{r}+\delta)\rho(\bar{z})\mathrm{d}\bar{r}\,\mathrm{d}\bar{z} \tag{13}$$

where

$$\begin{aligned}&\overline{S}_{rr}=\frac{\partial\bar{u}}{\partial\bar{r}},\quad \overline{S}_{\theta\theta}=\frac{1}{\bar{r}+\delta}(\bar{u}+s\bar{v}),\quad \overline{S}_{zz}=\frac{\partial\bar{w}}{\bar{\gamma}\partial\bar{z}},\\&\overline{S}_{r\theta}=-\frac{\bar{v}+s\bar{u}}{\bar{r}+\delta}+\frac{\partial\bar{v}}{\partial\bar{r}},\quad \overline{S}_{rz}=\frac{\partial\bar{w}}{\partial\bar{r}}+\frac{\partial\bar{u}}{\bar{\gamma}\partial\bar{z}},\\&\overline{S}_{\theta z}=\frac{\partial\bar{v}}{\bar{\gamma}\partial\bar{z}}-\frac{s\bar{w}}{\bar{r}+\delta},\quad \overline{E}_r=-\frac{\partial\bar{\varphi}}{\partial\bar{r}},\quad \overline{E}_\theta=\frac{s\bar{\varphi}}{\bar{r}+\delta},\\&\overline{E}_z=-\frac{\partial\bar{\varphi}}{\bar{\gamma}\partial\bar{z}},\quad \overline{H}_r=-\frac{\partial\bar{\psi}}{\partial\bar{r}},\quad \overline{H}_\theta=\frac{s\bar{\psi}}{\bar{r}+\delta},\quad \overline{H}_z=-\frac{\partial\bar{\psi}}{\bar{\gamma}\partial\bar{z}},\quad \bar{\gamma}=h/\overline{R}\\&C_{11}=\frac{E(z(\bar{z}))(1-v)}{(1+v)(1-2v)},\quad C_{12}=\frac{E(z(\bar{z}))v}{(1+v)(1-2v)},\quad C_{44}=\frac{E(z(\bar{z}))}{2(1+v)}\\&\Gamma_1=\int_0^{2\pi}\cos^2(s\theta)\mathrm{d}\theta=\begin{cases}2\pi & (s=0)\\ \pi & (s>0)\end{cases}\\&\Gamma_2=\int_0^{2\pi}\sin^2(s\theta)\mathrm{d}\theta=\begin{cases}0 & (s=0)\\ \pi & (s>0)\end{cases}\end{aligned} \tag{14}$$

Each of the displacement amplitude functions can be written as double series of Chebyshev polynomials multiplied by boundary functions, i.e.

$$\begin{aligned}&\overline{U}(\bar{r},\bar{z})=F_u^0(\bar{r})F_u^1(\bar{r})\sum_{i=1}^{I}\sum_{j=1}^{J}A_{ij}P_i(\bar{r})P_j(\bar{z})\\&\overline{V}(\bar{r},\bar{z})=F_v^0(\bar{r})F_v^1(\bar{r})\sum_{k=1}^{K}\sum_{l=1}^{L}B_{kl}P_k(\bar{r})P_l(\bar{z})\\&\overline{W}(\bar{r},\bar{z})=F_w^0(\bar{r})F_w^1(\bar{r})\sum_{m=1}^{M}\sum_{n=1}^{N}C_{mn}P_\mathrm{m}(\bar{r})P_n(\bar{z})\end{aligned} \tag{15}$$

where I, J, K, L, M and N are the truncation orders of the Chebyshev polynomial series, respectively; A_{ij}, B_{kl}, C_{mn}, D_{op} and E_{qh} are the coefficients to be determined; $P_i(\chi)$ $(i=1,2,\ldots;\ \chi=\bar{r},\bar{z})$ is the one-dimensional ith Chebyshev polynomial, i.e.