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Global solution curves for several classes of singular periodic problems

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ABSTRACT

Using continuation methods and bifurcation theory, we study the exact multiplicity of periodic solutions, and the global solution structure, for three classes of periodically forced equations with singularities, including the equations arising in micro-electro-mechanical systems (MEMS), the ones in condensed matter physics, as well as A.C. Lazer and S. Solimini's (Lazer and Solimini, 1987) problem.

1. Introduction

The study of periodic solutions of equations with singularities began with a remarkable paper of A.C. Lazer and S. Solimini [1]. One of the model problems considered in that paper involved (in case $c = 0$)

$$u''(t) + cu'(t) + \frac{1}{u^p(t)} = \mu + e(t), \quad u(t+T) = u(t), \tag{1.1}$$

where $e(t)$ is a continuous function of period T with $\int_0^T e(t)\,dt = 0$, $c \in R$, and $p > 0$. (Here and throughout the paper we write $u(t+T) = u(t)$ to indicate that we are searching for T-periodic solutions.) A.C. Lazer and S. Solimini [1] proved that the problem (1.1) has a positive T-periodic solution if and only if $\mu > 0$. It turned out later that problems with singularities occur often in applications. The recent book of P.J. Torres [2] contains a review of these applications, with up to date results, and a long list of open problems.

We approach the problem (1.1) with continuation (bifurcation) methods, and add some details to the Lazer–Solimini result, which turn out to be useful, in particular, for numerical computation of solutions. Write $u(t) = \xi + U(t)$, with $\int_0^T U(t)\,dt = 0$, so that ξ is the average of the solution $u(t)$. We show that ξ is a *global parameter* i.e., the value of ξ uniquely identifies both μ and the corresponding T-periodic solution $u(t)$ of (1.1). Hence, the set of positive T-periodic solutions of (1.1) can be represented by a curve in (ξ, μ)

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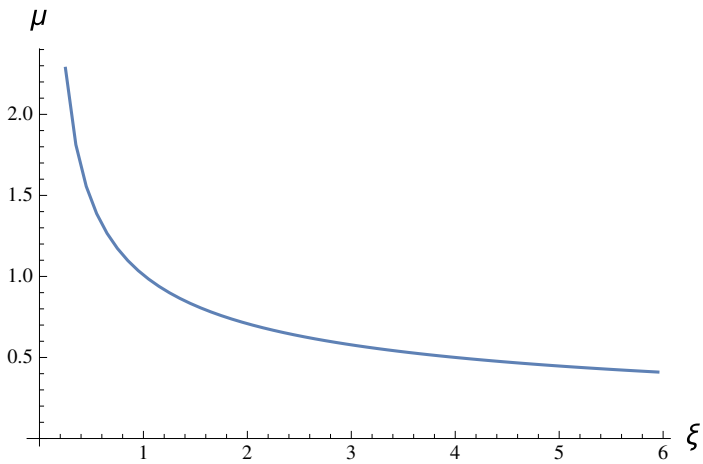

Fig. 1. An example for Theorem 4.1.

plane. We show that this curve, $\mu = \varphi(\xi)$, is hyperbola-like, where $\varphi(\xi)$ is a decreasing function, defined for all $\xi \in (0, \infty)$, and $\lim_{\xi\to 0} \varphi(\xi) = \infty$, $\lim_{\xi\to\infty} \varphi(\xi) = 0$. It turns out to be relatively easy to compute numerically the solution curve $\mu = \varphi(\xi)$, see Fig. 1. In the last section we explain the implementation of our numerical computations, using the *Mathematica* software. Using recent results of R. Hakl and M. Zamora [3], we also discuss the solution curve in case $e(t) \in L^2$.

There is a considerable recent interest in micro-electro-mechanical systems (MEMS), see e.g., J.A. Pelesko [4], Z. Guo and J. Wei [5], or P. Korman [6]. Recently A. Gutiérrez and P.J. Torres [7] have considered an idealized mass–spring model for MEMS, which they reduced to the following problem

$$u''(t) + cu'(t) + bu(t) + \frac{a(t)}{u^p(t)} = \mu + e(t), \quad u(t+T) = u(t), \tag{1.2}$$

with $p = 2$, and $e(t) = 0$ ($b > 0$ and $c \in R$ are constants). We show that the solution curve $\mu = \varphi(\xi)$ of (1.2) is parabola-like, and there is a $\mu_0 > 0$ so that the problem (1.2) has exactly two positive T-periodic solutions for $\mu > \mu_0$, exactly one positive T-periodic solution for $\mu = \mu_0$, and no positive T-periodic solutions for $\mu < \mu_0$, see Fig. 2. This extends the corresponding result in [7].

For both of the above equations we were aided by the fact that the nonlinearities were convex. An interesting problem where the nonlinearity changes convexity arises as a model for fluid adsorption and wetting on a periodic corrugated substrate:

$$u''(t) + cu'(t) + a\left(\frac{1}{u^4(t)} - \frac{1}{u^3(t)}\right) = \mu + e(t), \quad u(t+T) = u(t), \tag{1.3}$$

in case $c = 0$, and $\mu = 0$, see C. Rascón et al. [8], and also P.J. Torres [2], where this problem was suggested as an open problem. Using an idea from G. Tarantello [9], we again describe the shape of the solution curve, and obtain an exact multiplicity result, see Fig. 3. For the physically significant case when $c = 0$, and $\mu = 0$, our result implies the existence and uniqueness of positive T-periodic solution.

Remarkably, in all three cases the graph of the global solution curve is similar to that of the nonlinearity $g(u)$.

We now outline our approach. We embed (1.1) into a family of problems

$$u''(t) + cu'(t) + k\frac{1}{u^p(t)} = \mu + e(t), \quad u(t+T) = u(t), \tag{1.4}$$

with the parameter $0 \le k \le 1$. When $k = 0$ and $\mu = 0$, the problem is linear, and it has a unique T-periodic solution of any average ξ. We show that if the average of solutions is kept fixed, the solutions (u, μ) of (1.4)