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On a diffusion model with absorption and production

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ABSTRACT

We discuss the structure of radial solutions of some superlinear elliptic equations which model diffusion phenomena when both absorption and production are present. We focus our attention on solutions defined in $\mathbb{R}$ (regular) or in $\mathbb{R} \setminus \{0\}$ (singular) which are infinitesimal at infinity, discussing also their asymptotic behavior. The phenomena we find are present only if absorption and production coexist, i.e., if the reaction term changes sign. Our results are then generalized to include the case where Hardy potentials are considered.

1. Introduction

In this paper we are interested in structure results for radial solutions for a family of equations whose prototype has the following form

$$\Delta u(\mathbf{x}) + k(|\mathbf{x}|)u|u|^{q-2} = 0, \quad k(|\mathbf{x}|) = \begin{cases} K_1 & |\mathbf{x}| \le 1 \\ K_2 & |\mathbf{x}| > 1 \end{cases} \tag{L}$$

where $\mathbf{x} \in \mathbb{R}^n$, with $n > 2$, $q > 2$, $K_1 K_2 < 0$; we assume either $K_1 < 0 < K_2$ and $q > 2^*$, or $K_2 < 0 < K_1$ and $2_* < q < 2^*$, where $2_* := 2\frac{n-1}{n-2}$ and $2^* := \frac{2n}{n-2}$ are respectively the Serrin and the Sobolev critical exponent.

Since we just deal with radial solutions we will indeed consider the following singular ordinary differential equation

$$u'' + \frac{n-1}{r}u' + k(r)u|u|^{q-2} = 0, \quad k(r) = \begin{cases} K_1 & r \le 1, \\ K_2 & r > 1, \end{cases} \tag{Lr}$$

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where, abusing the notation, we have set $u(r) = u(\mathbf{x})$ for $|\mathbf{x}| = r$, and $'$ denotes differentiation with respect to r.

The interest in equations of the family (L) started long ago from nonlinearities where k is a constant, either negative or positive, and then it was generalized to include the case where k varies with r, thus finding several different possible structures for the solutions, see e.g. [1–11] for a far from being exhaustive bibliography. Nowadays it has become a broadly studied topic and the discussion now includes a wide family of non-linearities, see e.g. [12–15] and references therein. Radial solutions play a key role for (L), since in many cases, e.g. $k(r) \equiv K > 0$, positive solutions have to be radial (but also in many situations in which k is allowed to vary, see e.g. [3,13,16]). They are also crucial to determine the threshold between fading and blowing up initial data in the associated parabolic problem, see e.g. [17,18].

It can be shown that, when $q > 2$, positive solutions exhibit two behaviors as $r \to 0$ and as $r \to \infty$. In particular, $u(r)$ may be a *regular solution*, i.e. $u(0) = d > 0$ and $u'(0) = 0$, or a *singular solution*, i.e. $\lim_{r\to 0} u(r) = +\infty$, a *fast decay solution*, i.e. $\lim_{r\to\infty} u(r)r^{n-2} = L$, or a *slow decay solution*, i.e. $\lim_{r\to\infty} u(r)r^{n-2} = +\infty$.

Moreover, a regular, respectively singular, positive solution $u(r)$ defined for any $r > 0$ such that $\lim_{r\to\infty} u(r) = 0$ is usually called *ground states*, resp. *singular ground states*. In the whole paper we use the following notation: we denote by $u(r, d)$ the regular solution of (Lr) such that $u(0, d) = d$, and by $v(r, L)$ the fast decay solution such that $\lim_{r\to\infty} r^{n-2}v(r, L) = L$.

Eq. (L) is a widely studied topic and find many applications in different contexts. e.g., it can model the equilibria for a nonlinear heat equation. In this case u is the temperature and $ku|u|^{q-2}$ represents a thermo-regulated reaction which produces heat when $k > 0$, or absorbs heat when $k < 0$.

It can also model the equilibria reached by a series of chemical reactions, see e.g. [19] for a derivation of the model (in particular Chapter 7, and especially 7.4), see also [20, Section 1]. In this case u represents the density of a substance A reacting with substrates B and D according to the following scheme:

$$(q-1)A + B \underset{c_1}{\to} qA + C, \quad (q-1)A + D \underset{c_2}{\to} E. \tag{1.1}$$

In the first reaction c_1, we have $(q-1)$ particles A which react with some substrate B to produce C and a larger number of particles of A (say q in this case). In the second, c_2, we have $(q-1)$ particles A which react with some substrate D to produce E (in fact we obtain an equation of the same type also when the substrates B, D and the substance C are not present). The two reactions can be modeled respectively by

$$\begin{aligned} u_t &= \Delta u + \nu_1\mu_B\, u^{q-1}, \\ u_t &= \Delta u - \nu_2(q-1)\mu_D\, u^{q-1}, \end{aligned} \tag{1.2}$$

where ν_1 and ν_2 are the velocities of the reactions, and μ_B, μ_D are the densities of the substrates which are assumed to be constant (and they can be chosen to be 1). Here we are interested in the equilibria reached by u, assuming that we have diffusion (Δu), production inside a ball (e.g., when $K_1 = \nu_1\mu_B > 0$) and absorption outside (when $K_2 = -\nu_2(q-1)\mu_D < 0$), or the symmetric situation. Eq. (1.2) can model also a series of unknown reactions starting from the substances on the left of the arrow in (1.1) and ending with the ones on the right of the arrow: Usually in chemistry and especially in biochemistry we do not know all the intermediate steps which are actually taking place, so the models are constructed using just the starting reagents and the final products we find.

A further simple case (from the modeling viewpoint) we are able to deal with is the following type of reaction together with its inverse.

$$(q-1)A + B \underset{c_1}{\to} qA, \quad qA \underset{c_2}{\to} (q-1)A + B.$$