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Well-posedness of a nonlinear integro-differential problem and its rearranged formulation☆

Gonzalo Galiano[a,*], Emanuele Schiavi[b], Julián Velasco[a]

[a] Department of Mathematics, Universidad de Oviedo, c/ Calvo Sotelo, 33007-Oviedo, Spain
[b] Department of Mathematics, Universidad Rey Juan Carlos, 28993-Madrid, Spain

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ABSTRACT

We study the existence and uniqueness of solutions of a nonlinear integro-differential problem which we reformulate introducing the notion of the decreasing rearrangement of the solution. A dimensional reduction of the problem is obtained and a detailed analysis of the properties of the solutions of the model is provided. Finally, a fast numerical method is devised and implemented to show the performance of the model when typical image processing tasks such as filtering and segmentation are performed.

1. Introduction

This article is devoted to the study of the nonlinear integro-differential problem

$$\partial_t u(t,\mathbf{x}) = \int_\Omega \mathcal{K}_h(u(t,\mathbf{y}) - u(t,\mathbf{x}))(u(t,\mathbf{y}) - u(t,\mathbf{x}))d\mathbf{y} + \lambda(u_0(\mathbf{x}) - u(t,\mathbf{x})), \tag{1}$$

$$u(0,\mathbf{x}) = u_0(\mathbf{x}) \tag{2}$$

for $(t,\mathbf{x}) \in Q_T = (0,T) \times \Omega$. Here, $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ denotes an open and bounded set, $T > 0$, $\lambda > 0$ and $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$. The *range* kernel $\mathcal{K}_h$ is given as a rescaling $\mathcal{K}_h(\xi) = \mathcal{K}(\xi/h)$ of a kernel $\mathcal{K}$ satisfying the usual properties of nonnegativity and smoothness. We shall give the precise assumptions in Section 3. We shall refer to problem (1)–(2) as to problem $\mathrm{P}(\Omega, u_0)$. The main results contained in this article are:

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* Corresponding author. Tel.: +34 985103343; fax: +34 985103354.
E-mail addresses: galiano@uniovi.es (G. Galiano), emanuele.schiavi@urjc.es (E. Schiavi), julian@uniovi.es (J. Velasco).

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- Theorem 1. The well-posedness of problem $\mathrm{P}(\Omega, u_0)$, the stability property of its solutions with respect to the initial datum, and the time invariance of the level set structure of its solutions.
- Theorem 2. The equivalence between solutions of problem $\mathrm{P}(\Omega, u_0)$ and the one-dimensional problem $\mathrm{P}(\Omega_*, u_{0*})$, where $\Omega_* = (0, |\Omega|)$, and u_{0_*} is the decreasing rearrangement of u_0, see Section 2 for definitions.
- Theorem 3. The asymptotic behavior of the solution of problem $\mathrm{P}(\Omega_*, u_{0*})$ with respect to the window size parameter, h, as a shock filter.

Problem $\mathrm{P}(\Omega, u_0)$ is related to some problems arising in Image Analysis, Population Dynamics and other disciplines. The general formulation in (1) includes, for example, a time-continuous version of the Neighborhood filter (NF) operator:

$$\mathrm{NF}^h\, u(\mathbf{x}) = \frac{1}{C(\mathbf{x})} \int_\Omega \mathrm{e}^{-\frac{|u(\mathbf{x})-u(\mathbf{y})|^2}{h^2}} u(\mathbf{y}) d\mathbf{y},$$

where h is a positive constant, and $C(\mathbf{x}) = \int_\Omega \exp\left(-|u(\mathbf{x}) - u(\mathbf{y})|^2 h^{-2}\right) d\mathbf{y}$ is a normalization factor. In terms of the notation introduced for problem $\mathrm{P}(\Omega, u_0)$ the NF is recovered setting $\mathcal{K}(s) = \exp(-s^2)$ and $\lambda = 0$. This well known denoising filter is usually employed in the image community through an iterative scheme,

$$u^{(n+1)}(\mathbf{x}) = \frac{1}{C_n(\mathbf{x})} \int_\Omega \mathcal{K}_h(u^{(n)}(\mathbf{x}) - u^{(n)}(\mathbf{y}))u^{(n)}(\mathbf{y}) d\mathbf{y}, \tag{3}$$

with $C_n(\mathbf{x}) = \int_\Omega \mathcal{K}_h(u^{(n)}(\mathbf{x}) - u^{(n)}(\mathbf{y}))d\mathbf{y}$. It is the simplest particular case of other related filters involving nonlocal terms, notably the Yaroslavsky filter [1,2], the Bilateral filter [3,4], and the Nonlocal Means filter [5].

These methods have been introduced in the last decades as efficient alternatives to local methods such as those expressed in terms of nonlinear diffusion partial differential equations (PDE's), among which the pioneering nonlinear anti-diffusive model of Perona and Malik [6], the theoretical approach of Álvarez et al. [7] and the celebrated ROF model of Rudin et al. [8]. We refer the reader to [9] for a review comparing these local and non-local methods.

Another image processing task encapsulated by problem $\mathrm{P}(\Omega, u_0)$ is the *histogram prescription*, used for image contrast enhancement: Given an initial image u_0, find a companion image u such that u and u_0 share the same level sets structure, and the histogram distribution of u is given by a prescribed function Ψ. A widely used choice is $\Psi(s) = s$, implying that u has a uniform histogram distribution. In this case, $\mathcal{K}(s) = sign^-(s)/s$ and λ is related to the image size and its dynamic range, see Sapiro and Caselles [10] for the formulation and analysis of the problem. Nonlinear integro-differential of the form

$$\partial_t u(t, \mathbf{x}) = \int_\Omega (u(t, \mathbf{y}) - u(t, \mathbf{x}))w(\mathbf{x} - \mathbf{y})d\mathbf{y} \tag{4}$$

and other nonlinear variations of it have also been recently used (Andreu et al. [11]) to model diffusion processes in Population Dynamics and other areas. More precisely, if $u(t, \mathbf{x})$ is thought of as a density at the point $\mathbf{x}$ at time t and $w(\mathbf{x}-\mathbf{y})$ is thought of as the probability distribution of jumping from location $\mathbf{y}$ to location $\mathbf{x}$, then $\int_\Omega u(t, \mathbf{y})w(\mathbf{x}-\mathbf{y})d\mathbf{y}$ is the rate at which individuals are arriving at position $\mathbf{x}$ from all other places and $-u(t, \mathbf{x}) = -\int_\Omega u(t, \mathbf{x})w(\mathbf{x}-\mathbf{y})d\mathbf{y}$ is the rate at which they are leaving location $\mathbf{x}$. In the absence of external or internal sources this consideration leads immediately to the fact that the density u satisfies Eq. (4).

These kind of equations are called nonlocal diffusion equations since in them the diffusion of the density u at a point $\mathbf{x}$ and time t depends not only on $u(t, \mathbf{x})$ but also on the values of u in a set determined (and weighted) by the space kernel w. A thorough study of this problem may be found in the monograph by Andreu et al. [11]. Observe that in problem $\mathrm{P}(\Omega, u_0)$, the space kernel is taken as $w \equiv 1$, meaning that the influence of nonlocal diffusion is spread to the whole domain.

As noticed by Sapiro and Caselles [10] for the histogram prescription problem, and later by Kindermann et al. [12] for the iterative Neighborhood filter (3), or by Andreu et al. [11] for continuous time problems like