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Positive solutions of an asymptotically periodic Schrödinger–Poisson system with critical exponent☆

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ABSTRACT

Existence of one positive solution of the generalized Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u - K(x)\phi|u|^3u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)|u|^5 & \text{in } \mathbb{R}^3, \end{cases}$$

where V, K, f are asymptotically periodic functions of x, is proved by the mountain pass theorem and the concentration-compactness principle. The system with subcritical nonlocal term has been studied extensively in the last twenty years, while the system with critical nonlocal term has seldom been studied. It turns out that new techniques are needed in dealing with the case of critical nonlocal term.

1. Introduction and main result

In the last two decades, the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = g(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3 \end{cases} \tag{1.1}$$

was studied in numerous papers, due to the fact that solutions $(u(x), \phi(x))$ of (1.1) correspond to standing wave solutions $(e^{-i\lambda t}u(x), \phi(x))$ of the time-dependent system

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} = -\Delta\psi + \widetilde{V}(x)\psi + \phi\psi - \tilde{g}(x,|\psi|)\psi & \text{for } (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+, \\ -\Delta\phi = |\psi|^2 & \text{for } x \in \mathbb{R}^3, \end{cases}$$

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where i is the imaginary unit, $\mathbb{R}^+ = [0, \infty)$, $\widetilde{V}(x) = V(x) + \lambda$ and $\tilde{g}(x, |u|)u = g(x, u)$. Systems of this type stem from many physical problems, especially in quantum mechanics and in semiconductor theory [1–3]. In particular, (1.1) was introduced by Benci and Fortunato in [1] as a model describing standing waves for the nonlinear Schrödinger equations interacting with an unknown electrostatic field.

When the potential is radially symmetric or even a positive constant, many papers have been devoted to studying existence and multiplicity of nontrivial solutions of (1.1) under various assumptions on the nonlinearity (see e.g. [4–9]). In such a case, one can search solutions in the subspace of $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ which consists of radially symmetric functions. In the case where the potential is non-radially symmetric, (1.1) was also widely investigated (see e.g. [10–13]). With a periodic potential, (1.1) was studied in [13] and existence of infinitely many geometrically distinct solutions was proved by a superposition principle established in [14]. Very recently, a ground state solution of (1.1) was obtained in [15,16] where the potential is assumed to be an asymptotically periodic function. It is worth pointing out that the nonlinearity considered in [15,16] has subcritical growth and is independent of x. For more results of this system, we refer the reader to [17–30] and references therein. It should be mentioned that these aforementioned papers considered the Schrödinger–Poisson system with subcritical nonlocal term and, to the best of our knowledge, the Schrödinger–Poisson system with critical nonlocal term was only studied in [31,32].

Partially motivated by the works of [15,16,31,32], we are concerned with the generalized Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u - K(x)\phi|u|^3 u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)|u|^5 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where V, K, f are continuous functions. Such a system is related to the well known Choquard equation

$$-\Delta u + V(x)u - (I_\alpha * |u|^p)\,|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^3,$$

which was introduced as an approximation to the Hartree–Fock theory of one component plasma [33]. Here, $I_\alpha : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ denotes the Riesz potential and $p > 1$.

In order to state the main result of this paper, we denote by $\mathscr{B}$ the class of functions $b \in C(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ satisfying, for any $\varepsilon > 0$, the set $\{x \in \mathbb{R}^3 \mid |b(x)| \geq \varepsilon\}$ has finite Lebesgue measure. The assumptions on the potentials are formulated below. In (A_1) the potentials are assumed to be perturbations of periodic functions.

(A_1) There exist continuous and periodic functions $\hat{V}$ and $\hat{K}$, τ_i-periodic in x_i with $\tau_i > 0$, $i = 1, 2, 3$, such that $\hat{V} - V \in \mathscr{B}$, $K - \hat{K} \in \mathscr{B}$ and

$$0 < V_0 \leq V(x) \leq \hat{V}(x), \qquad 0 < \hat{K}(x) \leq K(x), \quad \text{for all } x \in \mathbb{R}^3.$$

(A_2) There exist $\bar{x} \in \mathbb{R}^3$ and $\alpha \in [1, 3)$ such that $K(\bar{x}) = \max_{x \in \mathbb{R}^3} K(x)$ and

$$K(x) = K(\bar{x}) + o(|x - \bar{x}|^\alpha) \quad \text{as } x \to \bar{x}.$$

Setting $F(x,t) = \int_0^t f(x,s)\,ds$, we introduce the following hypotheses on f.

(f_1) $f(x,t) = o(|t|)$ as $t \to 0$, uniformly for $x \in \mathbb{R}^3$.
(f_2) $\lim_{t\to\infty} f(x,t)/|t|^5 = 0$ and there exists $C > 0$ such that, for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$,

$$|f(x,t)| \leq C\left(1 + |t|^5\right).$$