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# Existence and stability of stationary solutions to the compressible Navier–Stokes–Poisson equations

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ABSTRACT

In this paper, we are concerned with the compressible Navier–Stokes–Poisson equations with the given external force of general form in three dimensional space. Based on the weighted $L^2$ method and the contraction mapping principle, we prove the existence and uniqueness of stationary solutions. Then, we show the stability of solutions to the Cauchy problem near the stationary state provided that the initial perturbation is sufficiently small. Finally, the time decay rates of the solutions are obtained when the initial perturbation belongs to $\dot{H}^{-s}$ with $0 \leq s < \frac{3}{2}$.



## 1. Introduction

In this paper, we consider the three-dimensional compressible Navier–Stokes–Poisson equations

$$\begin{cases} \partial_t \rho + \operatorname{div} m = 0, \\ \partial_t m + \operatorname{div}\left(\dfrac{m \otimes m}{\rho}\right) + \nabla P(\rho) = \mu \Delta \dfrac{m}{\rho} + (\mu + \lambda)\nabla \operatorname{div} \dfrac{m}{\rho} + \rho \nabla \Phi + \rho f, \\ \Delta \Phi = \rho - \bar{\rho}, \qquad \lim\limits_{|x| \to \infty} \Phi(t,x) = 0, \\ (\rho, m)(t,x)|_{t=0} = (\rho_0, m_0)(x). \end{cases} \tag{1.1}$$

Here $\rho(t,x) > 0, m(t,x), \Phi(t,x)$ represent the density, the momentum, the electrostatic potential of the electrons at time $t \geq 0$ and position $x \in \mathbb{R}^3$. The pressure function $P = P(\rho)$ is assumed to be a smooth function in a neighborhood of $\bar{\rho}$ satisfying $P'(\bar{\rho}) > 0$, where the constant $\bar{\rho} > 0$ denotes the background doping profile. The constants $\mu, \lambda$ are the viscosity coefficients with the usual physical conditions $\mu > 0, \lambda + \frac{2}{3}\mu \geq 0$. Moreover, $f(t,x)$ is a given external force.

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In comparison with the Navier–Stokes–Poisson equations studied in [1], where the external force is considered to be the potential force, the steady states of momentum in system (1.1) are no longer constant 0. The stationary problem corresponding to the initial value problem (1.1) is

$$\begin{cases} \operatorname{div} m_{st} = 0, \\ \operatorname{div}\left(\dfrac{m_{st} \otimes m_{st}}{\rho_{st}}\right) - \mu \Delta \dfrac{m_{st}}{\rho_{st}} - (\mu + \lambda)\nabla \operatorname{div} \dfrac{m_{st}}{\rho_{st}} + \nabla P(\rho_{st}) = \rho_{st} \nabla \Phi_{st} + \rho_{st} f, \\ \Delta \Phi_{st} = \rho_{st} - \bar{\rho}. \end{cases} \tag{1.2}$$

The compressible Navier–Stokes–Poisson equations (NSP) can be used to simulate, for instance in semiconductor devices, the transport of charged particles under the electric field of electrostatic potential force [2]. Recently, some important progress has made for the (NSP) system. For the pressure law $p(\rho) = \rho^\gamma$ with the adiabatic exponent $\gamma > 3/2$, the global existence of weak solutions was obtained by [3] when the spatial dimension is three in the framework of Lions and Feireisl for the compressible Navier–Stokes equations [4,5]. When there is no external force, we refer to [6–11]. Hao and Li [6], and Zheng [11] established the global strong solutions of the initial value problem for the multi-dimensional compressible (NSP) system in Besov space, respectively. The global existence and the optimal decay rates of the classical solution around a constant state were obtained by Li, Matsumura, and Zhang [7]. Wang and Wu [10] investigated the initial value problem for the (NSP) equations in $\mathbb{R}^n$ $(n \geq 3)$ and obtained the pointwise estimates of the solution by a detailed analysis of the Greens function to the corresponding linearized equations. Wang [9] observed the special construction of the (NSP) equations and posed some stronger conditions on the initial value and then proved the global existence and asymptotic decay of solutions in three dimensional space under smallness condition on the initial data. Recently, for the non-flat doping profile, Tan and his collaborators in [8] study the stability of the steady state of the compressible (NSP) equations, where they prove the global existence near the steady state for the large doping profile. From those work, a common feature shows that the momentum of the (NSP) system decays at the slower rate than that of the compressible Navier–Stokes system in the absence of the electric field, which thus implies that the electric field could affect the large time behavior of the solution and produce some additional difficulties of analysis.

When the external force is taken into consideration. Zhao and Li [1] studied the global existence and asymptotic behavior of smooth solution around the stationary solutions, while the external force is the potential force, i.e. $f = -\nabla\psi$, where $\psi$ is a scalar function, then there exists a unique stationary solution $(\tilde{\rho}, 0, \tilde{\Phi})$ if $\psi$ satisfies some smallness condition. When the external force is given by the general form $f = \operatorname{div} f_1 + f_2$, the stationary solution is nontrivial in general, for the compressible Navier–Stokes equations, cf. [12,13]. For the compressible Navier–Stokes–Korteweg equations, cf. [14]. But to the rate of convergence, Shibata and Tanaka in [15] obtained:

$$\|\nabla(\rho - \rho_*, u - u_*)\|_{H^2} \leq C(\kappa)(1+t)^{-\frac{1-\kappa}{2}},$$

for any small positive constant $\kappa$ when the initial perturbation belongs to $H^3(\mathbb{R}^3) \cap L^{6/5}(\mathbb{R}^3)$, but $C(\kappa)$ may become $\infty$ when $\kappa$ tends to zero. Notice that even when $\kappa = 0$, the rate is not optimal for the perturbation in the space $L^{6/5}$. For this reason, we study the optimal time decay rates of solutions when initial perturbation is in $H^{-s}$ with $0 \leq s < \frac{3}{2}$ in the last section. For more stability of nontrivial stationary solution, the interesting reader may refer to [16–20], for which we do not go into details here.

**Notations**: Throughout this paper, for simplicity, we will omit the variables $t, x$ of functions if it does not cause any confusion. $C$ denotes a generic positive constant which may vary in different estimates. We write $\|(A,B)\|_X := \|A\|_X + \|B\|_X$ and $\int g\,dx = \int_{\mathbb{R}^3} g\,dx$. Next, we introduce some function spaces. Let $H^m(\mathbb{R}^3), m \in \mathbb{Z}_+$ denote the usual $L^2$-Sobolev spaces with normal $\|\cdot\|_{H^m}$ and $L^p(\mathbb{R}^3), 1 \leq p \leq \infty$ be the usual $L^p$ spaces with norm $\|\cdot\|_{L^p}$. $\nabla^l$ with an integer $l \geq 0$ stands for the usual any spatial derivatives of order $l$. When $l < 0$ or $l$ is not a positive integer, $\nabla^l$ stands for $\Lambda^l$ defined by $\Lambda^l g := \mathcal{F}^{-1}(|\xi|^l \mathcal{F} g)$, where