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Adjoint symmetry and conservation law of nonlinear diffusion equations with convection and source terms

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ABSTRACT

A complete adjoint symmetry classification of the nonlinear diffusion equations with convection and source terms is performed and all adjoint symmetries are expressed in a unified form $X = \varphi(x,t)\partial_u$, where $\varphi(x,t)$ satisfies a linear partial differential equation. Moreover, we find that all the adjoint symmetries are conservation law multipliers of the equation under study. We also show that the adjoint symmetries are just the substitutions of nonlinear self-adjointness and vice versa. Finally, a general conservation law formula associated with the symmetry and adjoint symmetry is given and two illustrated examples are considered.

1. Introduction

The classical symmetry group theory provides an effective method to analyze partial differential equations (PDEs). For instance, it can be used to derive new solutions from known ones, classify PDEs to equivalent classes, linearize nonlinear PDEs to linear ones, construct conservation laws by Noether's theorem, etc. [1,2].

The classical method for finding symmetry of PDEs is the Lie group method of infinitesimal transformation and the associated symmetry determining system is linear and overdetermined. Roughly speaking, for the system of m PDEs of rth-order

$$E^\alpha(x,u,u_{(1)},\ldots,u_{(r)})=0,\quad \alpha=1,2,\ldots,m, \tag{1}$$

where $x=(x^1,\ldots,x^n)$ is an independent variable set and $u=(u^1,\ldots,u^m)$ is a dependent variable set, $u_{(i)}$ denotes all ith x derivatives of u, a symmetry $X_\eta=\eta^i(x,u,u_{(1)},\ldots,u_{(s)})\partial_{u^i}$ arises from solving the following system on the solution space of Eq. (1)

$$(\mathscr{L}_E)^\alpha_\rho\eta^\rho=\frac{\partial E^\alpha}{\partial u^\rho}\eta^\rho+\frac{\partial E^\alpha}{\partial u^\rho_{i_1}}D_{i_1}\eta^\rho+\cdots+\frac{\partial E^\alpha}{\partial u^\rho_{i_1\cdots i_r}}D_{i_1}\cdots D_{i_r}\eta^\rho=0. \tag{2}$$

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Note that, hereinafter, the summation convention for repeated indices is used and D_i denotes the total derivative operator with respect to x^i,

$$D_i = \frac{\partial}{\partial x^i} + u_i^\sigma \frac{\partial}{\partial u^\sigma} + u_{ij}^\sigma \frac{\partial}{\partial u_j^\sigma} + \cdots, \quad i = 1, 2, \ldots, n.$$

Since Lie group method is entirely algorithmic, thus many symbolic manipulation programs have been developed to facilitate the calculations [3]. Meanwhile, there have been several generalizations of the classical Lie symmetry which include non-classical symmetry, conditional symmetry and generalized symmetry and etc. [4–7].

An adjoint symmetry $X_\omega = \omega_\rho(x, u, u_{(1)}, \ldots, u_{(s)})\partial_{u^\rho}$ of system (1) is determined by the adjoint equations of system (2), i.e.,

$$(\mathscr{L}_E^*)_\alpha^\rho \omega_\rho = \omega_\rho \frac{\partial E^\rho}{\partial u^\alpha} - D_{i_1}\left(\omega_\rho \frac{\partial E^\rho}{\partial u^{\alpha i_1}}\right) + \cdots + (-1)^r D_{i_1} \cdots D_{i_r}\left(\omega_\rho \frac{\partial E^\rho}{\partial u^\alpha_{i_1 \cdots i_r}}\right) = 0. \tag{3}$$

However, comparing with the powerful functions of Lie symmetry, one cannot identify a sort of direct action of adjoint symmetry on the study of PDEs, but they do carry relevant information, in particular with respect to the conservation laws of PDEs [2].

Conservation laws have many significant effects, particularly with regard to integrability, constants of motion, and numerical solution methods, thus a number of methods are developed to construct conservation laws. For the PDEs admitting a variational principle, Noether's theorem gives a formula for obtaining local conservation laws by use of variational symmetries [1,2]. For the PDEs without having a variational principle, one may adopt direct method, partial Lagrangian method, multiplier method, nonlinear self-adjointness method to achieve the goal [8–12].

In particular, nonlinear self-adjointness provides an effective method to construct conservation laws of the system of PDEs whether it has a variational principle or not [13–15]. The main idea of the method is to turn the system of PDEs into Lagrangian equations by artificially adding additional variables, then to apply the theorem proved in [16] to construct local and nonlocal conservation laws. Meanwhile, the proposed conservation law formula only involves differential operation instead of integral operation and thus can be fully implemented on a computer. Approximate nonlinear self-adjointness and approximate conservation law of perturbed PDEs were considered in [12,17,18]. Quite recently, comparisons of nonlinear self-adjointness method with multiplier method are studied in [19,20].

This paper is devoted to study adjoint symmetry and conservation law of the nonlinear diffusion equations with convection and source terms

$$G = u_t - \big(F(u)u_x\big)_x - P(u)u_x - Q(u) = 0, \tag{4}$$

where $u = u(x, t)$ is an unknown function of two independent variables $x = x^1$ and $t = x^2$, $F(u)(F'(u) \neq 0), P(u)$ and $Q(u)$ are respectively three arbitrary differential functions and referred to as the diffusion, convection and source terms. A huge number of researches based on the symmetry-related methods are performed for Eq. (4) and its subclasses (see for examples [21–23] and references therein). Observe that Noether's theorem is not suitable for Eq. (4) since it is not derived from a variational principle. Thus conservation laws of certain subclasses of Eq. (4) were considered through direct method in [24,25] and references therein.

Eq. (4) generalizes numerous known nonlinear second-order evolution equations describing various processes in biology, ecology, physics, chemistry, and etc. [26–29]. For example, as an extension of linear heat equation $u_t = u_{xx}$, the well-known nonlinear heat equation

$$u_t = \big(F(u)u_x\big)_x, \tag{5}$$