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Reconstruction of noisy signals by minimization of non-convex functionals

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ABSTRACT

Non-convex functionals have shown sharper results in signal reconstruction as compared to convex ones, although the existence of a minimum has not been established in general. This paper addresses the study of a general class of either convex or non-convex functionals for denoising signals which combines two general terms for fitting and smoothing purposes, respectively. The first one measures how close a signal is to the original noisy signal. The second term aims at removing noise while preserving some expected characteristics in the true signal such as edges and fine details. A theoretical proof of the existence of a minimum for functionals of this class is presented. The main merit of this result is to show the existence of minimizer for a large family of non-convex functionals.

1. Introduction

Despite the many recent research efforts in the field of signal restoration, the design of robust solutions to the problem of signal denoising is still an open challenge. Traditionally, non-convex variational functionals have been successfully applied to this problem without any theoretical guarantee [1, Section 3.2.6].

The general denoising task can be modeled as a variational problem that consists in finding a global minimizer of a functional known as *energy* or *cost* function [1–4]. It is composed of two terms. The first one (the *fitting* or *fidelity* term) is intended to reward the closeness of a signal to the original noisy signal. The second one (the *potential* or *regularization* term) aims at removing noise while preserving some prior expected characteristics such as borders, details, among others. To this end, the potential term should favor the smoothing of regions with small or moderate gradients, usually corresponding to noises, while it should

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not penalize regions with strong gradients, usually corresponding to edges. The combination of these two terms can be understood as a trade-off between approximation accuracy and noise removal. Signals very close to the original noisy signal yield low cost values of the fitting term and high cost values in the regularization term. Otherwise, smooth signals can be substantially different from the original one, thus inducing a high cost value in the fitting term, but a low potential value.

There has been much work in the optimization of both convex and non-convex cost functionals. Convex functionals are very popular as they are easy to work with and they are guaranteed to have a minimizer. On the other hand, it has been systematically and empirically proved [1,5] that non-convex regularization provides more chances for image restoration with well-defined edges as compared to convex methods. However, the existence of a minimum for non-convex functionals cannot be established in general in continuous domains.

One of the pioneering works in the variational framework is a non-convex functional proposed by Geman and McClure [3], which has been extensively used for image restoration. After proving the non-existence of minimum for this functional, Chipot et al. [6] showed that perturbing it by adding a quadratic term yields a new non-convex functional with minimizer in one-dimensional signals, that also presents a good performance preserving edges. Formally, the proposal by Chipot et al. [6] was defined as follows:

$$E(u) = \int_a^b \left[u(x) - g(x)\right]^2 dx + \int_a^b \phi(u'(x))dx \tag{1}$$

where $\phi(t) = \frac{t^2}{\alpha^2+t^2} + \gamma t^2$.

The first term in (1) measures how much a function $u(x)$ fits the original noisy signal $g(x)$ in the least-squares sense, and it is particularly effective when additive Gaussian noise is present.

The second term (a regularization term) rewards more the smoothing of regions of moderate gradients (noise removal) than the smoothing of regions with large gradients (edge blurring). Therefore, it promotes clean signals with neat edges.

March et al. [7] proved the existence of a minimum for a generalized form of (1) in order to show the existence of a solution to the Perona–Malik equation [8] in the sense of BV [9]. This generalized form keeps unchanged the quadratic fitting term and introduces a general smooth robust function as a second term. However, this model does not consider non-smooth potentials which could provide better restoration results for certain problems. In this regard, Nikolova et al. [10] have shown that non-convex non-smooth potentials are better to preserve piecewise constant signals.

The works [6,7] guarantee the existence of a minimizer for certain non-convex denoising functionals in only one-dimensional signals in continuous domains. Recently, Harjulehto et al. [11] have demonstrated the existence of a minimum of the functional (1) on arbitrary dimensions, although this result does not hold for general functionals with arbitrary non-convex potential and fidelity terms. Non-convex fitting based on M-estimators [12] has shown to be quite robust in the presence of outliers caused by impulsive noise [13,14].

The two terms involved in the functionals proposed in [6,11,7] can be viewed as specific fitting and denoising solutions, respectively. In particular, the fidelity term has a quadratic form, which makes these functionals particular for Gaussian noise attenuation. However, these methods fail to remove Impulsive noise from classes of signals that arise naturally in various fields of applied sciences, for instance piecewise constant (PWC) and piecewise smooth (PWS) signals.

A theoretical study on the existence of minimizers for a family of non-convex functionals in the discrete setting is shown in [10], while [15] provides empirical evidence on the usefulness of a particular non-convex functional for image denoising and deblurring in the presence of impulsive noise. However, no proof on the existence of minimizers for nonconvex variational methods with both non-quadratic fidelity terms and edge-preserving potential terms in the continuous domain has been reported.