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Boundary blow-up rate and uniqueness of the large solution for an elliptic cooperative system of logistic type[☆]

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ABSTRACT

This paper ascertains the blow-up rates of each of the components of a singular boundary value problem related to a cooperative system of logistic type, in order to establish the uniqueness of the large solution. Astonishingly, the cooperative coupling does not change the blow-up rates of the uncoupled system provided these blow-up rates are sufficiently close, though it changes exactly one of them, keeping invariant the other, when they are bounded away. This seems to be the first time where this change of behavior has been documented in the literature.

1. Introduction

In this paper we study the uniqueness of the classical positive solution of the singular boundary value problem

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1 + a_{12} u_2 - \mathfrak{a}_1(x) u_1^{p_1} & \text{in } \Omega, \\ -\Delta u_2 = \lambda_2 u_2 + a_{21} u_1 - \mathfrak{a}_2(x) u_2^{p_2} & \\ u_1 = u_2 = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda_i \in \mathbb{R}$, $a_{ij} > 0$, $p_i > 1$, $i, j \in \{1, 2\}$, $N \in \mathbb{N}$, $N \geq 1$, Ω is an open subdomain of $\mathbb{R}^N$ of class $C^{2+\nu}$ for some $\nu \in (0, 1)$, and $\mathfrak{a}_i \in \mathcal{C}^{\nu}(\Omega)$ for $i \in \{1, 2\}$ satisfy $\mathfrak{a}_i(x) > 0$ for all $x \in \Omega$. The singular boundary condition should be understood in the sense that

$$\lim_{\operatorname{dist}(x,\partial\Omega)\downarrow 0} u_i(x) = +\infty, \quad i \in \{1, 2\}.$$

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Problem (1.1) is the simplest and most natural way of coupling the fully uncoupled problem

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1 - \mathfrak{a}_1(x) u_1^{p_1} & \text{in } \Omega, \\ -\Delta u_2 = \lambda_2 u_2 - \mathfrak{a}_2(x) u_2^{p_2} & \\ u_1 = u_2 = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

in a cooperative way, i.e., with $a_{12} > 0$ and $a_{21} > 0$, in order to have available the strong maximum principle for (1.1), [1,2]. The system in (1.1) goes back to [3–5]. When $a_{12}a_{21} < 0$, the system is of non-cooperative type. In such case, all the comparison techniques available for cooperative systems fail. In particular, the method of sub and supersolutions cannot be used. The case when $a_{12} < 0$ and $a_{21} < 0$ can be treated in a similar way, because, in such case, the system is quasi-cooperative (see, e.g., Chapter 10 of [6]).

Since (1.2) consists of two uncoupled singular boundary value problems for the logistic equation, which is the most paradigmatic one in population dynamics and mathematical biology, [6–8], the problem of analyzing the singular cooperative problem (1.1) should deserve a huge attention from the point of view of the applications. Actually, the solutions of (1.1) provide us with the asymptotic profiles of the solutions of wide classes of cooperative parabolic systems in the presence of spatial heterogeneities, [9,10,6].

Although there is a huge amount of literature devoted to the existence and uniqueness of large positive solutions for the single generalized logistic equation, as it becomes apparent by simply looking at [11–30], and the huge list of references in [6], and even there are some rather astonishing multiplicity results for large positive solutions, [31], the literature on systems seems substantially more reduced. More precisely, in [32] the existence of large solutions was characterized for the classical diffusive symbiotic model of Lotka–Volterra, and the blow-up rates of each of the components of these singular solutions were ascertained. In [9,10] a general version of (1.1) was analyzed. Although the existence of large solutions was established there in, the problem of their uniqueness, as well as the problem of ascertaining their blow-up rates, remained fully open. More recently, [33] established an optimal uniqueness result for a general class of radially symmetric cooperative systems with convex non-linearities. Finally, [34] proves the existence and uniqueness of large solutions for a class of *autonomous* reaction diffusion systems of cooperative type. Actually, Theorem 6 of [34] was inspiring to get one of our main results here, however the techniques developed in [34] cannot be applied to treat (1.1), by the presence of the spatial heterogeneities in (1.1).

The existence of solutions of (1.1) has been already established in [9]. The uniqueness for the radially symmetric counterpart in the spacial case $\lambda_1,\ \lambda_2 \geq 0$ was established by the authors in [33], where the uniqueness was inferred as an application of the maximum principle established in [1]. However, in this work the main uniqueness result relies on the fact that all positive solutions of (1.1) have identical blow-up rates on $\partial\Omega$. This important property, which follows from the localization method of [26], allows us to get uniqueness for a huge class of domains Ω, not necessarily radially symmetric. The next two results provide us with these blow-up rates.

Theorem 1.1. *Suppose that for each $i \in \{1,2\}$ there exist $\mathfrak{b}_i, \gamma_i \in \mathcal{C}(\partial\Omega)$, with $\mathfrak{b}_i(z) > 0$ for all $z \in \partial\Omega$, and $\gamma_i \geq 0$ on $\partial\Omega$, such that*

$$\lim_{\substack{x\to z\\ x\in\Omega,\, z\in\partial\Omega}} \frac{\mathfrak{a}_i(x)}{\mathfrak{b}_i(z)[\operatorname{dist}(x,\partial\Omega)]^{\gamma_i(z)}} = 1 \quad \textit{uniformly on } \partial\Omega. \tag{1.3}$$

Let $z \in \partial\Omega$ be such that

$$\frac{\gamma_1(z)+2}{p_1-1} + 2 - \frac{\gamma_2(z)+2}{p_2-1} > 0 \quad \textit{and} \quad \frac{\gamma_2(z)+2}{p_2-1} + 2 - \frac{\gamma_1(z)+2}{p_1-1} > 0. \tag{1.4}$$

Then, setting

$$\alpha_i(z) = \frac{\gamma_i(z)+2}{p_i-1}, \qquad A_i(z) = \left[\frac{\alpha_i(z)(\alpha_i(z)+1)}{\mathfrak{b}_i(z)}\right]^{\frac{1}{p_i-1}}, \quad i \in \{1,2\}, \tag{1.5}$$