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Solutions of perturbed Hammerstein integral equations with applications

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ABSTRACT

By means of topological methods, we provide new results on the existence, non-existence, localization and multiplicity of nontrivial solutions for systems of perturbed Hammerstein integral equations. In order to illustrate our theoretical results, we study some problems that occur in applied mathematics, namely models of chemical reactors, beams and thermostats. We also apply our theory in order to prove the existence of nontrivial radial solutions of systems of elliptic boundary value problems subject to nonlocal, nonlinear boundary conditions.

1. Introduction

Problems with *nonlinear* boundary conditions often occur in applied mathematics. For example, the fourth-order differential equation

$$u^{(4)}(t) = f(t, u(t)), \quad t \in (0, 1), \tag{1.1}$$

subject to the nonlinear boundary conditions (BCs)

$$u(0) = u'(0) = u''(1) = 0, \qquad u'''(1) = h(u(1)) \tag{1.2}$$

models the stationary states of the deflections of an elastic beam of length 1. The BCs (1.2) describe that the left end of the beam is clamped and the right end is free to move with a vanishing bending moment and a shearing force that reacts (in a possibly nonlinear manner) according to the displacement registered in the right end. Various methods were used to deal with the existence of solutions of the boundary value problem

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(BVP) (1.1)–(1.2), for example variational methods in [1,2], iterative methods in [3–5] and topological methods in [6].

One possibility is to rewrite this BVP as a *perturbed* Hammerstein integral equation, that is

$$u(t) = \gamma(t)h(u(1)) + \int_0^1 k(t,s)f(s,u(s))\,ds. \tag{1.3}$$

This kind of perturbed integral equation has been investigated in the past by a number of authors, we refer the reader to the manuscripts [3,7–18] and references therein.

When seeking the existence of *positive* solutions of the perturbed integral equation (1.3), typically one assumes either a *global* restriction on the growth of the nonlinearity h, say for example

$$\alpha_1 x \le h(x) \le \alpha_2 x, \quad \text{for every } x \ge 0, \tag{1.4}$$

where $0 \le \alpha_1 \le \alpha_2$, as in [14,6,19,20,15,21,22], or an *asymptotic* condition, as in [23,24,10,11,25,12,26,13], or a kind of *mixture* of the two, as in [17,18].

Our idea is to utilize a kind of *local* estimate on the growth of the nonlinearity h, that can be seen as a weakening of the global assumption (1.4). This approach is useful under two points of view: it allows to handle a wider class of nonlinearities with respect to the assumption (1.4) and is convenient in order to prove multiplicity results, henceforth improving and complementing the above results.

We stress that we can deal with *nonlocal* BCs; for example we can replace the BCs (1.2) with

$$u(0) = u'(0) = u''(1) = 0, \qquad u'''(1) = h(u(\eta)),$$

where $\eta \in (0,1)$. This models a feedback mechanism where the shearing force in the right end of the beam reacts to the displacement registered in a point η. As far as we know the study of nonlocal BCs, in the context of ODEs, can be traced back to Picone [27], who considered multi-point BCs. For an introduction to nonlocal problems we refer the reader to the reviews [28–32] and the papers [33–35].

In Section 2 we discuss the existence of solutions of the more general equation

$$u(t) = \gamma(t)H[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,ds, \tag{1.5}$$

where H is a suitable compact functional in the space of continuous functions. We investigate the existence of *strictly positive*, *non-negative* and *nontrivial* solutions of (1.5), depending on the *sign properties* of the kernel k. This kind of equation is fairly general and can be applied to a variety of problems. As an *example* we apply our results in the case of three mathematical models, widely studied in literature, namely a chemical reactor, a cantilever beam and a thermostat model. We also present *non-existence* results for (1.5). In order to illustrate our approach to the reader, in Section 2 we restrict our attention to the case of *one* compact perturbation of the Hammerstein integral equation.

In Section 3 we further develop the methodology of the previous section and we deal with the case of systems of two perturbed Hammerstein equations. Here we focus, for brevity, to the problem of the existence of multiple, *nontrivial* solutions of the system

$$\begin{aligned} u(t) &= \sum_{j=1,2} \gamma_{1j}(t)H_{1j}[u,v] + \int_0^1 k_1(t,s)g_1(s)f_1(s,u(s),v(s))\,ds, \\ v(t) &= \sum_{j=1,2} \gamma_{2j}(t)H_{2j}[u,v] + \int_0^1 k_2(t,s)g_2(s)f_2(s,u(s),v(s))\,ds, \end{aligned} \tag{1.6}$$

where H_{ij} are compact functionals. Some non-existence results for (1.6) are also presented. Our approach allows us to deal with a wide class of systems of differential equations subject to *nonlinear* nonlocal BCs. As