Nonlinear Analysis: Real World Applications 30 (2016) 32–40

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

On the uniqueness for the 2D MHD equations without magnetic diffusion

Renhui Wan

School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China

ARTICLE INFO

Article history:
Received 11 March 2015
Received in revised form 26 October 2015
Accepted 25 November 2015
Available online 9 December 2015

Keywords:
MHD equations
Uniqueness
Magnetic diffusion

ABSTRACT

In this paper, we obtain the uniqueness of the 2D MHD equations, which fills the gap of recent work by Chemin et al. (2015).

1. Introduction

This paper considers the 2D MHD equations given by

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p - \nu\Delta u = B\cdot\nabla B, \\ \partial_t B + u\cdot\nabla B - B\cdot\nabla u = 0, \\ \mathrm{div} u = 0, \qquad \mathrm{div} B = 0, \\ u(x,0) = u_0(x), \qquad B(x,0) = B_0(x). \end{cases} \tag{1.1}$$

Here $t \geq 0$, $x \in \mathbb{R}^2$, $u = u(x,t)$ and $B = B(x,t)$ are vector fields representing the velocity and the magnetic field, respectively, $p = p(x,t)$ denotes the pressure and ν is a positive viscosity constant.

(1.1) has been investigated by many mathematicians. In 2014, by establishing a generalized Kato–Ponce estimate (see [1] for the well-known result):

$$\langle u\cdot\nabla B \mid B\rangle_{\dot{H}^s} \leq C\|\nabla u\|_{H^s}\|B\|_{H^s}^2, \quad s > \frac{d}{2},\ d = 2, 3,$$

Fefferman et al. [2] obtained the local existence and uniqueness for (1.1) and related models with the initial data $(u_0, B_0) \in H^s(\mathbb{R}^d),\ s > \frac{d}{2}$. For other results concerning regularity criteria, we refer to [3,4].

E-mail addresses: rhwanmath@zju.edu.cn, rhwanmath@163.com.

http://dx.doi.org/10.1016/j.nonrwa.2015.11.006
1468-1218/

Very recently, Chemin et al. in [5] obtained the local existence for (1.1) in 2D and 3D. But for the 2D case, the uniqueness was not obtained. Our main result is filling the gap of their works. The details can be described as follows:

Theorem 1.1. *For* $u_0 \in B^0_{2,1}(\mathbb{R}^2)$ *and* $B_0 \in B^1_{2,1}(\mathbb{R}^2)$ *with* $\mathrm{div} u_0 = \mathrm{div} B_0 = 0$, *there exists a time* $T = T(\nu, \|u_0\|_{B^0_{2,1}}, \|B\|_{B^1_{2,1}}) > 0$ *such that the system* (1.1) *has a unique solution* (u, B) *with*

$$u \in C([0,T]; B^0_{2,1}(\mathbb{R}^2)) \cap L^1([0,T]; B^2_{2,1})$$

and

$$B \in C([0,T]; B^1_{2,1}(\mathbb{R}^2)).$$

2. Preliminaries

Let $\mathfrak{B} = \{\xi \in \mathbb{R}^d,\ |\xi| \le \frac{4}{3}\}$ and $\mathfrak{C} = \{\xi \in \mathbb{R}^d,\ \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$. Choose two nonnegative smooth radial function $\chi,\ \varphi$ supported, respectively, in $\mathfrak{B}$ and $\mathfrak{C}$ such that

$$\chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d,$$
$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

We denote $\varphi_j = \varphi(2^{-j}\xi),\ h = \mathfrak{F}^{-1}\varphi$ and $\tilde{h} = \mathfrak{F}^{-1}\chi$, where $\mathfrak{F}^{-1}$ stands for the inverse Fourier transform. Then the dyadic blocks Δ_j and S_j can be defined as follows:

$$\Delta_j f = \varphi(2^{-j}D)f = 2^{jd}\int_{\mathbb{R}^d} h(2^j y) f(x-y)dy,$$
$$S_j f = \sum_{k\le j-1} \Delta_k f = \chi(2^{-j}D)f = 2^{jd}\int_{\mathbb{R}^d} \tilde{h}(2^j y) f(x-y)dy.$$

Formally, $\Delta_j = S_j - S_{j-1}$ is a frequency projection to annulus $\{\xi :\ C_1 2^j \le |\xi| \le C_2 2^j\}$, and S_j is a frequency projection to the ball $\{\xi :\ |\xi| \le C2^j\}$. One can easily verify that with our choice of φ

$$\Delta_j \Delta_k f = 0 \quad \text{if } |j-k| \ge 2 \quad \text{and} \quad \Delta_j(S_{k-1} f \Delta_k f) = 0 \quad \text{if } |j-k| \ge 5.$$

With the introduction of Δ_j and S_j, let us recall the definition of the Besov space.

Let $s \in \mathbb{R}$, $(p,q) \in [1,\infty]^2$, the homogeneous space $\dot{B}^s_{p,q}$ is defined by

$$\dot{B}^s_{p,q} = \{f \in \mathfrak{S}';\ \|f\|_{\dot{B}^s_{p,q}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{sqj} \|\Delta_j f\|^q_{L^p}\right)^{\frac{1}{q}}, & \text{for } 1 \le q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{sj} \|\Delta_j f\|_{L^p}, & \text{for } q = \infty, \end{cases}$$

and $\mathfrak{S}'$ denotes the dual space of $\mathfrak{S} = \{f \in \mathcal{S}(\mathbb{R}^d);\ \partial^\alpha \hat{f}(0) = 0;\ \forall\, \alpha \in\ \mathbb{N}^d$ multi-index$\}$ and can be identified by the quotient space of $\mathcal{S}'/\mathcal{P}$ with the polynomials space $\mathcal{P}$.

Let $s > 0$, and $(p,q) \in [1,\infty]^2$. The inhomogeneous Besov space $B^s_{p,q}$ is defined by

$$B^s_{p,q} = \{f \in \mathcal{S}'(\mathbb{R}^d);\ \|f\|_{B^s_{p,q}} < \infty\},$$