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Stability behaviors of Leray weak solutions to the three-dimensional Navier–Stokes equations

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ABSTRACT

This paper is devoted to the investigation of stability behaviors of Leray weak solutions to the three-dimensional Navier–Stokes equations. For a Leray weak solution of the Navier–Stokes equations in a critical Besov space, it is shown that the Leray weak solution is uniformly stable with respect to a small perturbation of initial velocity and external forcing. If the perturbation is not small, the perturbed weak solution converges asymptotically to the original weak solution as the time tends to the infinity. Additionally, an energy equality and weak–strong uniqueness for the three-dimensional Navier–Stokes equations are derived. The findings are mainly based on the estimations of the nonlinear term of the Navier–Stokes equations in a Besov space framework, the use of special test functions and the energy estimate method.

1. Introduction

In this study we consider the Cauchy problem of the three-dimensional (3D) incompressible Navier–Stokes equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = f, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x) \end{cases} \tag{1.1}$$

in the whole space $\mathbb{R}^3$, where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ denotes the unknown velocity field, p presents the unknown pressure field, f is an external force and $u_0(x)$ is an initial velocity field.

In the pioneer work [1], Leray obtained the existence of a global weak solution $u(x,t)$ to the 3D Navier–Stokes equations for $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. However, the regularity and uniqueness of

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the weak solution remain fundamentally unknown (refer to [2,3]). In order to understand the regularity and uniqueness of the Leray weak solution, Leray [1] posed the question of determining whether or not a weak solution of the Navier–Stokes equations (1.1) with $f = 0$ converges to zero in the L^2 norm as time tends to infinity. This problem has been investigated extensively by many authors and is referred to as a time decay problem (see [4] and references therein). It should be mentioned that the time decay problem can be renamed as the asymptotically stable problem of the trivial solution $u = 0$. It is natural and important to consider the stability issue of nontrivial solutions of the Navier–Stokes equations (1.1) when the forcing $f(t)$ is not small for large t. More precisely, we consider the following 3D perturbed Navier–Stokes equations

$$\begin{cases} \partial_t v - \Delta v + (v\cdot\nabla)v + \nabla\pi = f + g, \\ \nabla\cdot v = 0, \\ v(x,0) = u_0(x) + w_0(x), \end{cases} \tag{1.2}$$

where $w_0(x)$ is a perturbed velocity field and $g(x,t)$ is a perturbed forcing. The investigation of the stability behaviors including both uniform stability and asymptotic stability is beneficial to the understanding of regularity and uniqueness of the Leray weak solution to the 3D Navier–Stokes equations.

For the uniform stability of Navier–Stokes equations, Ponce et al. [5] considered the Leray weak solution (finite energy solution) u of the zero-forced Navier–Stokes equations in the critical space

$$\nabla u \in L^4(0,\infty; L^2(\mathbb{R}^3))$$

and proved that if

$$\|w_0\|_{H^1} + \int_0^\infty \left(\|g(t)\|_{L^2} + \|g(t)\|_{L^2}^2\right) dt \le \delta$$

for the sufficiently small constant $\delta > 0$, then there is a unique solution $v(t)$ of the perturbed Navier–Stokes equations (1.2) such that

$$\sup_{t>0} \|u(t) - v(t)\|_{H^1(\mathbb{R}^3)} \le M(\delta),$$

where $M(\delta)$ is a constant satisfying $\lim_{\delta\to 0} M(\delta) = 0$. Gui and Zhang [6] recently made an important improvement on the uniform stability for a weak solution of the horizontal viscous Navier–Stokes equations under the anisotropic Sobolev spaces $C(0,\infty; H^{0,s}(\mathbb{R}^3))$. Gallagher and Planchon [7] obtained the uniform stability for a weak solution $u(x,t)$ of n-dimensional small perturbed Navier–Stokes equations when u is subject to the assumption

$$u \in L^p(0,T; \dot{B}^{n/q+2/p-1}_{q,p}(\mathbb{R}^n)), \quad \frac{n}{q} + \frac{2}{p} > 1, \quad 2 < q,\ p < \infty. \tag{1.3}$$

It should be mentioned that there is still a gap between the solution in Gallagher–Planchon [7] and Cannone–Meyer–Planchon's solution [8] since the summability index in Besov space $p < \infty$. It seems difficult to fill with this gap by applying the Gallagher–Planchon's methods since their argument strictly depends on the consistency of the time integrability index p and the summability index in Besov space p. The first purpose of this study is to investigate the uniform stability problem for the finite energy weak solutions of three-dimensional Navier–Stokes equations with respect to the limiting case $p = \infty$ of the summability index in Besov space of Gallagher–Planchon's result [7] (see Theorem 1.1).

It is worthy noting that the uniform stability for the regular solutions with infinite energy has recently been studied by many authors. One may refer to Kawanago [9] for Kato's solution, Auscher et al. [10] for Koch–Tataru's solution, and Chemin and Gallagher [11] (see also [12]) for $\mathcal{C}^{-1}$ solution. These solutions essentially fall into the concept of scaled solutions introduced by Chemin [13]. Let us make it precise. Introduce the following scaling:

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x) \quad \text{for } \lambda > 0.$$