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Fringing field can prevent infinite time quenching

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ABSTRACT

Consider the following problem arising in Micro-Electro-Mechanical Systems (MEMS)

$$\begin{cases} u_t - \Delta u = \dfrac{\lambda(1+\delta \mid \nabla u \mid^2)}{(1-u)^p}, & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ 0 \leqslant u_0(x) < 1, & x \in \Omega, \end{cases}$$

where $\delta > 0$, $p > 1$ and Ω is a bounded smooth domain in $\mathbb{R}^N (N \geqslant 1)$. We prove that infinite time quenching is impossible for any $\lambda > 0$ in this problem. It provides a remarkable contrast to the case of $\delta = 0$, in which infinite time quenching must happen for some λ when Ω is a ball in $\mathbb{R}^N (N \geqslant 8)$. This means that the presence of the fringing field $\delta|\nabla u|^2$ dramatically changes the quenching behavior of the solution. We also obtain some new results about global convergence and quenching in finite time.

1. Introduction

Consider the following nonlinear parabolic problem

$$\begin{cases} u_t - \Delta u = \dfrac{\lambda(1+\delta|\nabla u|^2)}{(1-u)^p}, & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad 0 \leqslant u_0(x) < 1, & x \in \Omega, \end{cases} \tag{1.1}$$

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where $\lambda > 0$, $\delta > 0$, $p > 1$, Ω is a bounded smooth domain in $\mathbb{R}^N (N \geqslant 1)$ and $u_0 \in H_0^1(\Omega)$. The corresponding stationary problem is

$$\begin{cases} -\Delta u = \dfrac{\lambda(1+\delta|\nabla u|^2)}{(1-u)^p}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

modeling an electrostatically actuated Micro-Electro-Mechanical System (MEMS) device. In this model, λ represents the applied voltage, the term $\delta|\nabla u|^2$ describes the effect of a fringing electrostatic field, and δ is the aspect ratio of the device. About details, we refer to [1,2]. Notice that $p = 2$ (the inverse square relation in Coulomb's Law) and the physically relevant dimensions – $N = 1, 2$ – are the main concerns of MEMS researchers.

In this paper, we are concerned with quenching phenomena for problem (1.1). The solution of (1.1) is said to *quench* at time T if $\max_{\overline{\Omega}} u(x,t) \to 1$ as $t \to T$. If $T = +\infty$, then u quenches in *infinite* time; if $T < +\infty$, then u quenches in *finite* time (see e.g. [3–7]).

We recall some related results about (1.1) and (1.2) with $\delta = 0$. When Ω is a ball of $\mathbb{R}^N (N \geqslant 2)$, (1.2) has been well investigated by Joseph and Lundgren [8] (also see Fila et al. [9] and Zheng [10]); for $N = 1$, see Laetsch [11] or Levine [7]. A striking feature is the dependence of bifurcation diagrams on the dimension of space (see e.g. Fig. 1). The critical value λ^* of λ, also referred to as the *extremal parameter*, depends on p and Ω. λ^* is also intimately associated with quenching phenomena of (1.1). Acker and Walter [3] proved, under our notations, that for $u_0(x) \equiv 0$ and Ω is a ball, if $\lambda < \lambda^*$, the solution of (1.1) globally exists, while if $\lambda > \lambda^*$, the solution must quench in finite time; these conclusions are still true for $p = 2$ and general bounded domains (see [12]). For $\lambda = \lambda^*$, the situation is more complicated. Fila et al. [9] gave a complete answer for $u_0(x) \equiv 0$ and Ω being a ball (also see [10] for $0 \leqslant u_0(x) < 1$). They proved, under our notations, that infinite-time quenching occurs at λ^* if and only if $3 \leqslant N \leqslant 9$ and $0 < p \leqslant \frac{N-2\sqrt{N-1}}{4-N+2\sqrt{N-1}}$ or $N > 9$. For example, when $p = 2$, the critical value of N is $\frac{14+6\sqrt{6}}{3} \approx 7.933$, this means that when $\lambda = \lambda^*$, infinite-time quenching occurs just for all dimensions $N \geqslant 8$; see Fig. 1 for a comparison. In [4], Fila and Kawohl showed that when $\lambda = \lambda^*$ and Ω is a bounded convex domain, quenching in infinite time cannot occur if $N = 2$, $p > 1$ or $N = 3$, $p > 3$. For $p = 2$ and general bounded domains Ω, Esposito et al. [12] proved that when $\lambda = \lambda^*$, infinite time quenching cannot occur for $N \leqslant 7$; on the other hand, if $N \geqslant 8$ and the domain Ω is not a ball, the situation is still unclear. Besides, when $2 \leqslant N \leqslant 7$ and $\lambda = \lambda_*$ (see Fig. 1), the possibility of infinite time quenching is still not excluded for general initial data.

We now come back to the case $\delta > 0$. For $N = 1$, bifurcation diagrams of stationary problem (1.2) are similar to those of $\delta = 0$; see Fila et al. [13] for $p = 1$ and the authors [14] for general $p > 0$ (see Fig. 2). For $N \geqslant 2$, Wei and Ye [15] proved an interesting result that when $p > 1$, there exists a number $\lambda^* > 0$ such that (1.2) has at least two solutions for $\lambda \in (0, \lambda^*)$, exactly one solution for $\lambda = \lambda^*$, and no solutions for $\lambda \in (\lambda^*, +\infty)$; also see [1,2] for more related results of numerical simulations and [16] for another proof. For $N = 1$ and $p = 2$, (1.1) was investigated by Liu and Wang [17] and the results were similar to those when $\delta = 0$. Recently, Wang [18] considered a slightly more general problem than (1.1) and proved that for $u_0(x) \equiv 0$, if $\lambda < \lambda^*$, the solution converges pointwise to the minimal stationary solution; if $\lambda > \lambda^*$, the solution quenches in finite time. In a very recent work, Luo and Yau [22] gave another proof for Wang's result on Eq. (1.1), covering the case $\lambda = \lambda^*$.

Notice that Wei and Ye's result implies that the extremal solution of (1.2) at λ^* is always classical and the regularity is independent of the dimension N. It is natural to conjecture that there exists a counterpart in the corresponding parabolic problem (1.1). Namely, whether the quadratic gradient term $\delta|\nabla u|^2$ (i.e. the fringing field) can remove the effect of the dimension on quenching at λ^*?

The purpose of this paper is to give an affirmative answer for this problem. We prove that for any $\lambda > 0$ and $0 \leqslant u_0 < 1$, infinite time quenching cannot occur for problem (1.1). In other words, the solution u