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Blow-up of nonradial solutions to attraction–repulsion chemotaxis system in two dimensions

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ABSTRACT

This paper deals with the attraction–repulsion chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w), & x\in\Omega,\ t>0,\\ 0 = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0, \end{cases}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain in $\mathbb{R}^2$. We study the finite-time blowup of nonradial solutions in the parameter values $\chi\alpha - \xi\gamma > 0$ and $\beta \neq \delta$.

1. Introduction

In this paper, we consider the following attraction–repulsion chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w), & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ \tau w_t = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^2$. This model (1.1) was posed in [1] to describe the aggregation of microglia observed in Alzheimer's disease. The movement of microglia is directed by the interaction of

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chemoattractant (e.g., β-amyloid) and chemorepellent (e.g., TNF-α). In (1.1), $u(x,t)$ denotes the density of microglia, and $v(x,t)$ and $w(x,t)$ represent chemical concentration of attractant and repellent, respectively. The positive parameters χ, ξ describe the chemosensitivity, while $\alpha, \beta, \gamma, \delta$ show the chemical production and degradation rates.

As a subsystem, (1.1) contains a prototypical model by setting $w = 0$, as introduced by Keller and Segel [2] given by

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v + \alpha u - \beta v, & x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

which has been studied extensively for several decades. Many efforts have been made to study problems related to the topics of global existence, blowup and asymptotic behavior, see [3,4] for instance. Among others, the finite-time blowup of solutions is a striking indication of the aggregation phenomena of cells due to chemotaxis. It turned out that the mathematical difficulties linked to the subtle task of finding unbounded solutions can significantly be reduced upon setting $\tau = 0$ (cf.[5–7]). In two dimensions, Jäger and Luckhaus [8] first proved that if Ω is a disk, there exists $c^* > 0$, such that if $\chi\alpha \cdot \int_\Omega u_0(x)dx > c^*$, then the explosion happens in the center of the disk in finite time. Nagai [6,7] improved their result by determining $c^* = 8\pi/(\chi\alpha)$ for the radial and nonradial cases. For fully parabolic Keller–Segel system, no progress has been made until a recent work by Winkler [9] where it is shown that in higher dimensions, there exists radially symmetric solution blowing up in finite time with proper initial conditions.

In the case $\beta \neq \delta$, the study concerning blowup or global existence on attraction–repulsion chemotaxis system (1.1) is much harder due to the lack of a Lyapunov functional. However, the model (1.1) has been studied by some authors, see [10–15] for instance and the references therein. In one dimension, traveling plateaus for a hyperbolic Keller–Segel system with attraction and repulsion were studied in [16]. In [11] among other results, Liu and Wang studied the global existence of solutions and steady states of solutions to (1.1) with $\tau = 1$ and $n = 1$. In [12], Tao and Wang studied the global solvability, boundedness and blowup of solutions to the system (1.1) in the case $\tau = 0$. When $\beta \neq \delta$ and $\tau = 1$, they established the time dependent global boundedness of solutions. Recently, the boundedness for the fully parabolic attraction–repulsion model in the two dimensional case was obtained in [14,17]. Among others, they left an open question of establishing the finite-time blowup of solutions to (1.1) with $\tau = 0$ or 1 and $\beta \neq \delta$. Quite recently, Espejo and Suzuki [18, Proposition 3] obtained the finite-time blowup for radial solutions in a disk $B(0,R) \subset \mathbb{R}^2$ for $\tau = 0$, $\chi\alpha - \xi\gamma > 0$ and β, δ arbitrary positive constants.

Since the blow-up of nonradial solutions for $\beta \neq \delta$ to (1.1) is still open, our aim in the present paper is to obtain the conditions about finite-time blowup for nonradial solutions of (1.1) with $\tau = 0$ in a bounded domain $\Omega \subset \mathbb{R}^2$. The present paper extends the well-known blow-up result [7] to a new one for the attraction–repulsion chemotaxis system (1.1). Compared to [7], the technical obstacle in this paper lies in the fact that $\beta \neq \delta$ in (1.1). The novel point is to estimate the new integral term II_1 in (3.16) in Section 3.

Our main result reads as follows:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and let $x_0 \in \Omega$ be fixed. Assume that $\int_\Omega u_0(x) |x - x_0|^2 dx$ is sufficiently small, then if either of the following cases holds,*

(i) $\chi\alpha - \xi\gamma > 0$, $\delta \geq \beta$ *and* $\int_\Omega u_0(x)dx > \frac{8\pi}{\chi\alpha - \xi\gamma}$;

(ii) $\chi\alpha\delta - \xi\gamma\beta > 0$, $\delta < \beta$ *(which guarantees* $\chi\alpha - \xi\gamma > 0$*) and* $\int_\Omega u_0(x)dx > \frac{8\pi\beta}{\chi\alpha\delta - \xi\gamma\beta}$,

then the corresponding solution of (1.1) blows up in finite time.