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Blowup criteria for strong solutions to the compressible Navier–Stokes equations with variable viscosity

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ABSTRACT

For the compressible Navier–Stokes equations with viscosity and heat conductivity coefficients possibly depending on the density or temperature, several blowup criteria are given to the local-in-time strong solutions. The proof is based on energy methods together with elliptic and parabolic estimates adopted to the present situation.

1. Introduction

In continuum fluid mechanics, a widely accepted model to describe the evolution of compressible viscous fluids with heat conduction is the following well-known Navier–Stokes–Fourier system:

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \operatorname{div} \mathbb{S}, \tag{1.2}$$

$$\partial_t(\varrho E) + \operatorname{div}(\varrho E \mathbf{u}) + \operatorname{div}(\mathbf{q} - \mathbb{S}\mathbf{u} + p\mathbf{u}) = 0. \tag{1.3}$$

Here $\varrho, \mathbf{u}$ and E are the density, velocity and total energy of the fluid respectively. The system represents the conservation of mass, momentum and energy in the absence of the external force and heat production. The total energy density E is the sum of the kinetic and internal part, i.e.,

$$E = \frac{1}{2}|\mathbf{u}|^2 + e$$

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with e the specific internal energy. In terms of e, Eq. (1.3) is formulated as

$$\partial_t(\varrho e) + \mathrm{div}(\varrho e \mathbf{u}) + \mathrm{div}\, \mathbf{q} = \mathbb{S} : \nabla \mathbf{u} - p\mathrm{div}\, \mathbf{u}. \tag{1.4}$$

The stress tensor $\mathbb{S}$, according to Newton's rheological law, is given by

$$\mathbb{S} = \nu \left(\nabla \mathbf{u} + \nabla^t \mathbf{u} - \frac{2}{3} \mathrm{div}\, \mathbf{u}\mathbb{I} \right) + \eta \mathrm{div}\, \mathbf{u}\mathbb{I} = 2\nu d(\mathbf{u}) + \lambda \mathrm{div}\, \mathbf{u}\mathbb{I}. \tag{1.5}$$

Here we denote

$$d(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla^t \mathbf{u}), \qquad \lambda = \eta - \frac{2}{3}\nu.$$

The heat flux $\mathbf{q}$ obeys Fourier's law, i.e.,

$$\mathbf{q} = -\kappa \nabla \vartheta. \tag{1.6}$$

We assume that the Lamé viscosity coefficients $\nu = \nu(\varrho, \vartheta), \eta = \eta(\varrho, \vartheta)$ and heat conduction coefficient $\kappa = \kappa(\vartheta)$ are smooth functions of ϱ or ϑ satisfying

$$\nu(\varrho, \vartheta) \geq \underline{\nu} > 0, \qquad \eta(\varrho, \vartheta) \geq 0, \qquad \kappa(\vartheta) \geq \underline{\kappa} > 0 \quad \text{for } \varrho, \vartheta \geq 0. \tag{1.7}$$

Given ϱ and ϑ, the pressure p and the internal energy e are determined by the equations of state:

$$p = p(\varrho, \vartheta), \qquad e = e(\varrho, \vartheta),$$

which satisfy the thermodynamic stability conditions:

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \qquad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0 \quad \text{for } \varrho, \vartheta > 0.$$

For smooth solutions, Eq. (1.3) can be also replaced by the following entropy equation:

$$\partial_t(\varrho s) + \mathrm{div}(\varrho s \mathbf{u}) + \mathrm{div} \frac{\mathbf{q}}{\vartheta} = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla \mathbf{u} - \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta} \right) - p\mathrm{div}\, \mathbf{u}. \tag{1.8}$$

Here $s = s(\varrho, \vartheta)$ is the *entropy* determined from the Gibbs' equation:

$$\vartheta \mathrm{d}s = \mathrm{d}e + p\mathrm{d}\left(\frac{1}{\varrho} \right).$$

For simplicity we consider ideal fluids, that is

$$p = A\varrho\vartheta, \qquad e = c_v \varrho \vartheta, \quad A, c_v > 0, \tag{1.9}$$

where A is the ideal gas constant and c_v is the specific heat. In this case system (1.1)–(1.3) reads as

$$\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \tag{1.10}$$

$$\varrho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla p = \mathrm{div}\, \mathbb{S}, \tag{1.11}$$

$$c_v \varrho(\partial_t \vartheta + \mathbf{u} \cdot \nabla \vartheta) - \mathrm{div}(\kappa \nabla \vartheta) = \mathbb{S} : \nabla \mathbf{u} - p\mathrm{div}\, \mathbf{u}. \tag{1.12}$$

System (1.10)–(1.12) is supplemented with initial data

$$\varrho|_{t=0} = \varrho_0(x), \qquad \mathbf{u}|_{t=0} = \mathbf{u}_0(x), \qquad \vartheta|_{t=0} = \vartheta_0(x), \tag{1.13}$$

where $x \in \Omega$, $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$, or a bounded smooth domain in $\mathbb{R}^3$. In the last case we impose the Dirichlet boundary conditions on the velocity field,

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{1.14}$$