Nonlinear Analysis: Real World Applications 28 (2016) 1–19

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

# On the Schrödinger–Poisson system with a general indefinite nonlinearity

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ARTICLE INFO

*Article history:*
Received 9 June 2015
Accepted 6 September 2015
Available online 29 September 2015

*Keywords:*
Non-autonomous
Schrödinger–Poisson system
Variational method
Positive solutions

ABSTRACT

We study the existence and multiplicity of positive solutions of a class of Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + u + l(x)\phi u = k(x)g(u) + \mu h(x)u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $k \in C(\mathbb{R}^3)$ changes sign in $\mathbb{R}^3$, $\lim_{|x|\to\infty} k(x) = k_\infty < 0$, and the nonlinearity $g$ behaves like a power at zero and at infinity. We mainly prove the existence of at least two positive solutions in the case that $\mu > \mu_1$ and near $\mu_1$, where $\mu_1$ is the first eigenvalue of $-\Delta + id$ in $H^1(\mathbb{R}^3)$ with weight function $h$, whose corresponding positive eigenfunction is denoted by $e_1$. An interesting phenomenon here is that we do not need the condition $\int_{\mathbb{R}^3} k(x)e_1^p dx < 0$, which has been shown to be a sufficient condition to the existence of positive solutions for semilinear elliptic equations with indefinite nonlinearity (see e.g. Costa and Tehrani, 2001).



## 1. Introduction

This is a continuation of our recent work [1], in which we studied the existence of multiple positive solutions to the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + l(x)\phi u = k(x)|u|^{p-2}u + \mu h(x)u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

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http://dx.doi.org/10.1016/j.nonrwa.2015.09.001
1468-1218/

where $4 < p < 6$, $l \in L^2(\mathbb{R}^3)$, $k \in C(\mathbb{R}^3)$ changes sign in $\mathbb{R}^3$ and $\lim_{|x|\to\infty} k(x) = k_\infty < 0$. In [1] we mainly proved that system (1.1) has at least two positive solutions for $\mu \geq \mu_1$ (but not far from $\mu_1$), where $\mu_1$ is the first eigenvalue of $-\Delta + id$ in $H^1(\mathbb{R}^3)$ with weight function $h$, whose corresponding eigenfunction is denoted by $e_1$. An interesting phenomenon there is that we have succeeded in making use of the nonlocal term to technically help deal with the key difficulty that the indefinite nonlinearity has created, and we do not need the condition

$$\int_{\mathbb{R}^3} k(x) e_1^p dx < 0, \tag{$*$}$$

which has been shown to be a sufficient condition to the existence of positive solutions for semilinear elliptic equations with indefinite nonlinearities with a bounded or an unbounded domain, like

$$-\Delta_m u = \lambda a(x)|u|^{m-2}u + k(x)f(u), \quad 1 < m < N,$$

where $-\Delta_m u = \mathrm{div}(|\nabla u|^{m-2}\nabla u)$ is $m$-Laplacian, $k$ is sign changing, and $f$ behaves like $|u|^{p-2}u$ near zero, and $e_1 \in D^{1,m}(\mathbb{R}^3)$ is the eigenfunction corresponding to the first eigenvalue of the eigenvalue problem $-\triangle_m u = \lambda a(x)|u|^{m-2}u$ in $D^{1,m}(\mathbb{R}^3)$, see Alama–Tarantello [2], Costa–Tehrani [3], Drábek–Huang [4] and their references therein. In Alama–Tarantello [2], where the authors consider the indefinite problem in a bounded domain, condition $(*)$ is shown to be necessary for the homogeneous $f$. In this work, instead of considering the homogeneous nonlinearity $k(x)|u|^{p-2}u$, we are concerned with a more general nonlinearity $k(x)g(u)$, where $g$ is a nonlinear function with superquadratic growth both at zero and at infinity. Surprisingly, we find that, for this general case, it is still not necessary to involve the condition $(*)$ either. Moreover, comparing with the results in [1], where $l \in L^2(\mathbb{R}^3)$, we allow $l \in L^\infty(\mathbb{R}^3)$ in the present paper. This also extends the results in [1], because when $l \in L^2(\mathbb{R}^3)$, the functional $\int_{\mathbb{R}^3} l(x)\left(\frac{1}{|x|} * (lu^2)\right)u^2(x)dx$ is weakly continuous; while for $l \in L^\infty(\mathbb{R}^3)$, the functional $\int_{\mathbb{R}^3} l(x)\left(\frac{1}{|x|} * (lu^2)\right)u^2(x)dx$ may be not weakly continuous.

More precisely, in the present paper, we study the positive solutions to the following problem

$$\begin{cases} -\Delta u + u + l(x)\phi u = k(x)g(u) + \mu h(x)u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = l(x)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where for the continuous nonlinearity $g \in C(\mathbb{R}, \mathbb{R})$, we assume the hypotheses $(G)$:

$(G_1)$ there is $q \in \mathbb{R}$ with $4 < q < 6$ such that $\lim_{s\to 0} \frac{g(s)}{|s|^{q-2}s} = 1$;
$(G_2)$ there is $p \in \mathbb{R}$ with $4 < p < 6$ such that $\lim_{|s|\to\infty} \frac{g(s)}{|s|^{p-2}s} = 1$;
$(G_3)$ $g(s) > 0$ for all $s > 0$.

Since we just aim to find the positive solutions, it is only necessary to consider all $u > 0$ for problem (1.2), and throughout the paper we assume, without loss of generality, that $g$ is defined in $\mathbb{R}$ as an odd function. For the weight functions we consider the following hypotheses $(H)$:

$(H_h)$ $h \in L^{3/2}(\mathbb{R}^3)$, $h(x) \geq 0$ for any $x \in \mathbb{R}^3$ and $h \not\equiv 0$;
$(H_h)$ $k \in C(\mathbb{R}^3)$ and $k$ changes sign in $\mathbb{R}^3$;
$(H_h)$ $\lim_{|x|\to\infty} k(x) = k_\infty < 0$;
$(H_h)$ $l \in L^\infty(\mathbb{R}^3)$, $l(x) \geq 0$ for any $x \in \mathbb{R}^3$ and $l \not\equiv 0$;
$(H_h)$ $l = 0$ a.e. in $\Omega^0$, where $\Omega^0 = \{x \in \mathbb{R}^3 : k(x) = 0\}$.

Our main result is as follows:

**Theorem 1.1.** *Suppose the hypotheses* $(G)$ *and* $(H)$. *In addition, if one of the following conditions holds:*