Nonlinear Analysis: Real World Applications 28 (2016) 48–71

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Asymptotic behavior of pulsating fronts and entire solutions of reaction–advection–diffusion equations in periodic media

Zhen-Hui Bu[a], Zhi-Cheng Wang[a,*], Nai-Wei Liu[b]

[a] School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, People's Republic of China

[b] School of Mathematics and Information Science, Yantai University, Yantai, Shandong 264005, People's Republic of China

ARTICLE INFO

Article history:
Received 4 November 2014
Received in revised form 4 August 2015
Accepted 16 September 2015
Available online 11 November 2015

Keywords:
Reaction–advection–diffusion equations
Periodic media
Pulsating fronts
Asymptotic behavior
Entire solution

ABSTRACT

This paper is concerned with the reaction–advection–diffusion equations with bistable nonlinearity in periodic media. Assume that the equation has three equilibria: an unstable equilibrium θ and two stable equilibria 0 and 1. It is known that there exist different pulsating fronts connecting any two of those three equilibria. In this paper we first study the exponential behavior of the fronts when they approach their stable limiting states. Then, we establish three different types of pulsating entire solutions for the equation. To establish the existence of entire solutions, we consider combinations of any two of those different pulsating fronts and construct appropriate sub- and supersolutions.

1. Introduction

In this paper, we investigate the following reaction–advection–diffusion equation

$$u_t - \nabla \cdot (A(x)\nabla u) + q(x) \cdot \nabla u = f(x,u), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.1}$$

Let $\mathcal{C}$ be the periodic cell defined by

$$\mathcal{C} = \{x \in \mathbb{R}^N \mid x \in (0,L_1) \times \cdots \times (0,L_N)\}$$

for some $(L_i)_{1\le i\le N} \in (\mathbb{R}^+)^N$. A function $v(x)$ defined in $\mathbb{R}^N$ is L-periodic with respect to the variable x if there holds

$$v(x+k) = v(x) \quad \text{for all } x \in \mathbb{R}^N \quad \text{and} \quad k \in L_1\mathbb{Z} \times \cdots L_N\mathbb{Z}.$$

* Corresponding author.
E-mail address: wangzhch@lzu.edu.cn (Z.-C. Wang).

http://dx.doi.org/10.1016/j.nonrwa.2015.09.006
1468-1218/

Now, we give some assumptions on the coefficients. The diffusion matrix $A(x) = (A_{ij}(x))_{1\le i,j\le N}$ is assumed to be a symmetric $C^3(\overline{\Omega})$ matrix and such that

$$\begin{cases} \exists\ 0 < c_1 \le c_2,\ \forall\ \xi \in \mathbb{R}^N,\ \forall\ x \in \mathbb{R}^N, \quad c_1|\xi|^2 \le \Sigma_{1\le i,j\le N} A_{ij}(x)\xi_i\xi_j \le c_2|\xi|^2, \\ \forall\ 1 \le i \le N,\ 1 \le j \le N, \quad A_{ij} \text{ is } L\text{-periodic w.r.t. } x. \end{cases} \tag{1.2}$$

Then, there exists a constant $\mathcal{R}$ such that

$$\|A\| \le \mathcal{R} \quad \text{and} \quad |\alpha A \beta^T| \le \mathcal{R}|\alpha|\,|\beta|, \quad \forall\ \alpha, \beta \in \mathbb{R}^N. \tag{1.3}$$

The advection coefficient $q(x) = (q_i(x))_{1\le i\le N}$ is of class $C^{1,\delta}(\mathbb{R}^N)$ for some $\delta \in (0,1)$ and such that

$$\begin{cases} \nabla \cdot q = 0 \quad \text{in } \mathbb{R}^N, \\ \forall\ 1 \le i \le N, \quad q_i \text{ is } L\text{-periodic w.r.t. } x, \\ \forall\ 1 \le i \le N, \quad \displaystyle\int_{\mathcal{C}} q_i(x)dx = 0. \end{cases} \tag{1.4}$$

Let the nonlinearity $f(x,u)$ be L-periodic w.r.t. x and of class $C^{1,\delta}(\mathbb{R}^N \times \mathbb{R})$ such that

$$\begin{cases} \forall\ x \in \mathbb{R}^N, \quad f(x,0) = f(x,1) = 0, \\ \exists\ \theta \in (0,1), \qquad f(x,\theta) = 0, \qquad f(x,u) < 0 \quad \text{for } u \in (0,\theta) \quad \text{and} \quad f(x,u) > 0 \quad \text{for } u \in (\theta,1), \\ \exists\ \rho_1 \in (0,1),\ \forall\ x \in \mathbb{R}^N, \quad f'_u(x,0)u \le f(x,u) \text{ for } u \in (0,\rho_1), \\ f(x,u) \le f'_u(x,1)(u-1) \quad \text{for } u \in (1-\rho_1, 1), \\ \displaystyle\int_{\mathcal{C}\times[0,1]} f(x,u)dxdu > 0. \end{cases} \tag{1.5}$$

Then f is a bistable nonlinearity. An archetype of such a function f has the form $f(x,u) = u(1-u)(u-\theta)$. Another archetype is $f(x,u) = u(1-u)(u-\theta)h(x)$, where $h(x)$ is a positive function and is L-periodic w.r.t. x.

Remark 1.1. By the assumption (1.5), it is easy to see that there exist a constant $K > 0$ and a sufficiently small constant $0 < \rho < \rho_1$ such that

$$\begin{cases} \left|f(x,\eta) - \xi^0(x)\eta\right| \le K\eta^{1+\delta} \quad \text{for } 0 < \eta < \rho,\ x \in \mathcal{C}, \\ \left|f(x,1-\eta) - \xi^1(x)\eta\right| \le K\eta^{1+\delta} \quad \text{for } 0 < \eta < \rho,\ x \in \mathcal{C}. \end{cases} \tag{1.6}$$

In particular, one has

$$f_u(x,u) < 0 \quad \text{for all } (x,u) \in \mathbb{R}^N \times [0,\rho) \cup (1-\rho, 1].$$

Define

$$\xi^0(x) = f'_u(x,0), \qquad \xi^1(x) = f'_u(x,1), \qquad \xi^\theta(x) = f'_u(x,\theta).$$

Let μ^i $(i = 0,\theta,1)$ denote the principal eigenvalue of the following linearized operator

$$\begin{cases} \mathcal{L}_0^i\psi := -\nabla \cdot (A(x)\nabla\psi) + q(x)\nabla\psi - \xi^i(x)\psi, \\ \psi \text{ is } L\text{-periodic w.r.t. } x \end{cases}$$

in the sense that there exist positive functions φ^i $(i = 0,\theta,1)$ in $\mathbb{R}^N$ such that

$$\begin{cases} \mathcal{L}_0^i\varphi^i = \mu^i\varphi^i \quad \text{in } \mathbb{R}^N, \\ \varphi^i \text{ is } L\text{-periodic w.r.t. } x. \end{cases}$$

Then by the assumption (1.5), one can easily get

$$\mu^0 > 0 \quad \text{and} \quad \mu^1 > 0.$$