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A Finsler geodesic spray paradigm for wildfire spread modelling

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ABSTRACT

One of the finest and most powerful assets of Finsler geometry is its ability to model, describe, and analyse in precise geometric terms an abundance of physical phenomena that are genuinely asymmetric, see e.g. Antonelli et al. (1993, 2003), Yajima and Nagahama (2009), Bao et al. (2004), Cvetič and Gibbons (2012), Gibbons et al. (2007), Astola and Florack (2011), Caponio et al. (2011), Yajima and Nagahama (2015). In this paper we show how *wildfires* can be naturally included into this family. Specifically we show how the celebrated and much applied Richards' equations for the large scale elliptic wildfire spreads have a rather simple Finsler-geometric formulation. The general Finsler framework can be explicitly 'integrated' to provide detailed – and curvature sensitive – geodesic solutions to the wildfire spread problem. The methods presented here stem directly from first principles of 2-dimensional Finsler geometry, and they can be readily extracted from the seminal monographs Shen (2001) and Bao et al. (2000), but we will take special care to introduce and exemplify the necessary framework for the implementation of the geometric machinery into this new application — not least in order to facilitate and support the dialog between geometers and the wildfire modelling community. The 'integration' part alluded to above is obtained via the geodesics of the ensuing Finsler metric which represents the local fire templates. The 'paradigm' part of the present proposal is thus concerned with the corresponding shift of attention from the actual fire-lines to consider instead the geodesic spray – the 'fire-particles' – which together, side by side, mould the fire-lines at each instant of time and thence eventually constitute the local and global structure of the wildfire spread.

1. Introduction

Every day the World is confronted with wildfires in various regions of our globe. Any wildfire is a highly nonlinear phenomenon, which is in pertinent demand for multidisciplinary and multi-scale analysis and better understanding. Detailed understanding is needed — both for emergency planning, which depends severely on quick and reliable predictions of the wildfire spread in time, as well as for the proper understanding of global issues concerning the CO_2 releases and biological and physical changes to the land surface [1]. Such phenomena obviously present scientific opportunities with no shortage of social significance. This fact is

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repeatedly stressed and documented in every paper that is concerned with the understanding, predicting, and modelling of wildfires, see e.g. [2]. Correspondingly there are several explicit and recent calls from the fire fighter community for new appropriate and effective first principles, i.e. new mathematical models, to handle and understand better the spreading mechanism of the wildfires in forests, grasslands, and wheat fields — with wind, slope, varying fuel properties across the domain and in geographically complicated terrain, see for example the description of the wildfire simulator Prometheus in [3], the comparison of various simulators in [4,5], and the general surveys as in e.g. [6–8].

As already alluded to in the abstract, Finsler geometry is a very strong tool for modelling physical phenomena that are genuinely asymmetric and/or non-isotropic, see e.g. [9–17]. In this paper we show how the geometric analysis of wildfires can be naturally added to this long list of applications of Finsler geometry.

1.1. Outline of paper

We briefly describe the standard modelling of wildfires including Huyghens' principle in Section 2. In Section 3 we emphasize and illustrate how to set up a general fire template field in a parameter domain. The principles of Finsler metrics, the ensuing first variation of arc-length, and the important notion of F-geodesics are surveyed in Sections 4 and 5. The resulting F-geodesic spray, its enveloping properties, and the induced exponential wildfires are constructed in Sections 6 and 7. In Sections 8 and 9 the Richards' equations are discussed in terms of their Randers–Zermelo equivalents, and we show that for elliptic wildfires the Richards' equations are solved by the corresponding Finsler-geodesic sprays. Specific examples of F-geodesic spray driven wildfires are constructed and illustrated in Sections 10–12. The final two Sections 13 and 14 present the main conclusions from the present paper together with a brief suggestion for further work.

2. Huyghens' principle

Following the pioneering works of G. D. Richards [18–22], van Wagner [23], Anderson et al. [24], and Glasa–Halada [25–29], we will apply a number of assumptions to be satisfied by the wildfires. We will only consider 2-dimensional, regular, smooth and deterministic wildfires ignited at time $t = 0$ on a smooth and regular ignition fireline (or at an ignition point). The fire spread is then represented by a smooth and regular vector function $\gamma(s,t)$ in a (u,v)-parameter domain $\mathcal{U} \subset \mathbb{R}^2$ so that $\gamma(s,0) = \eta_0(s)$ (the initial fireline) and so that $\gamma(s,t) = \eta_t(s)$ is the smooth and regular fireline at time $t > 0$. In particular — as part of this assumption we stop the fire before it creates singularities, cut-loci or bear-hugs. In this sense the analysis presented here is only semi-global, but, as we shall see, several global aspects follow naturally already from this outset.

Moreover, we assume that the linearized local spread profile, the so-called firelet, from every point in the fuel domain is known and that it is modelled by a time-invariant and strongly convex oval with the ignition point marked in its interior. This *pointed oval field* is eventually to be considered as the so-called indicatrix field (see the precise formal Definition 4.1) for the ensuing Finsler metric via which the wildfires are moulded and spread.

Remark 2.1. Time-invariance of the indicatrix field is a strong and not quite realistic condition to assume. Although wildfires usually spread relatively fast the fuel conditions in a given region will clearly change significantly during just 24 h. We refer to [19,30] for the first attempts to incorporate time-dependent fuel data and meteorological conditions into the enveloping method for elliptic indicatrix fields. (The latter reference seems, however, to build on a mis-interpretation of the first mentioned reference.) In the setting of [19] the elliptic fuel data are only allowed to vary as given functions of time — they are not allowed to vary spatially. We claim that the present Finsler geodesic spray paradigm, that will be unfolded below, can