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Set-valued equilibrium problems with applications to Browder variational inclusions and to fixed point theory

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ABSTRACT

In this paper, we deal with set-valued equilibrium problems under mild conditions of continuity and convexity on subsets recently introduced in the literature. We obtain that neither semicontinuity nor convexity are needed on the whole domain when solving set-valued and single-valued equilibrium problems. As applications, we derive some existence results for Browder variational inclusions, and we extend the well-known Berge maximum theorem in order to obtain two versions of Kakutani and Schauder fixed point theorems.

1. Introduction

On the trail of Browder's study of variational inclusions [1], many authors have been interested in inclusions involving set-valued mappings, see for instance [2–6]. In recent years, the notion of set-valued equilibrium problem has been employed in [3,5] in connection with the so-called equilibrium problem or inequality of Ky Fan-type (see [7–9]) which has produced an abundance of results in various areas of mathematics.

Let C be a nonempty subset of a (suitable) Hausdorff topological space and $\Phi : C \times C \rightrightarrows \mathbb{R}$ a set-valued mapping. Following [3,5], a *set-valued equilibrium problem* is a problem of the form

$$\text{find } x^* \in C \text{ such that } \Phi(x^*, y) \subset \mathbb{R}_+ \quad \forall y \in C. \tag{SVEP}$$

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We will also consider in the paper the following weaker set-valued equilibrium problem

$$\text{find } x^* \in C \text{ such that } \Phi(x^*, y) \cap \mathbb{R}_+ \neq \emptyset \quad \forall y \in C. \tag{SVEP(W)}$$

Recall that the so-called equilibrium problem is a problem of the form

$$\text{find } x^* \in C \text{ such that } \varphi(x^*, y) \geq 0 \quad \forall y \in C \tag{EP}$$

where $\varphi : C \times C \to \mathbb{R}$ is a bifunction.

It is well known that several problems arising in nonlinear analysis such as variational inequality problems, optimization problems, inverse optimization problems, mathematical programming, complementarity problems, fixed point problems and Nash equilibrium problems are special cases of equilibrium problems, see [10–12,2,13,14,3–6] and the references therein. As already mentioned in the literature, it is worth recalling that one of the interest of the equilibrium problem is that it unifies, at least, all the above mentioned problems in a common formulation and many techniques and methods established in order to solve one of them may be extended, with suitable adaptations, to equilibrium problems.

In the investigation about solving equilibrium problems, it has been considered recently in [15,16,10–12] the notion of hemicontinuity and semicontinuity on a subset. Various results on the existence of solutions of equilibrium problems have been obtained without the hemicontinuity and the semicontinuity of the bifunction on the whole domain, but just on the set of coerciveness.

On the other hand, a notion of self-segment-dense subset has been introduced in [5] allowing the authors to obtain some generalizations of the results of [3] on set-valued equilibrium problems.

In this paper, we deal mainly with set-valued equilibrium problems. We introduce and develop some notions of semicontinuity of set-valued mappings on a subset where the beginning notions have been first considered in [15,16,10–12]. We give some characterizations of lower and upper semicontinuity of set-valued mappings on a subset by means of lower and upper inverse sets in order to establish our results rather than using generalized sequences (nets).

In Section 2, we present the notions of semicontinuity of extended real valued functions and the semicontinuity of set-valued mappings, and give some preliminary results we need in the sequel. We also recall the necessary background on the subject such as the Ky Fan lemma and the notion of KKM mapping. The notion of self-segment-dense subset and its related results are also given.

In Section 3, we deal with single-valued and set-valued equilibrium problems and obtain, under these mild conditions of semicontinuity and convexity, existence results for the above three equilibrium problems considered in the paper.

In Section 4, we apply our results to Browder variational inclusions and obtain as a corollary an existence result for the well-known Browder–Hartman–Stampacchia variational inequality problems. We also give a generalization to the Berge maximum theorem and apply it to carry out two versions of Kakutani and Schauder fixed point theorems.

2. Notations and preliminary results

In this section we give the necessary background related to continuity and convexity of functions and set-valued mappings we need in the paper. We also establish some characterizations and preliminary results which will play a key role in the sequel.

In all the paper, $\mathbb{R} =]-\infty, +\infty[$ denotes the set of real numbers and $\overline{\mathbb{R}} = [-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\}$. We also make use of the following notation: $\mathbb{R}_+ = [0, +\infty[$, $\mathbb{R}_+^* =]0, +\infty[$, $\mathbb{R}_- = -\mathbb{R}_+$ and $\mathbb{R}_-^* = -\mathbb{R}_+^*$.

Let X be a Hausdorff topological space. Recall that an extended real valued function $f : X \to \overline{\mathbb{R}}$ is said to be *lower semicontinuous* at $x_0 \in X$ if for every $\epsilon > 0$, there exists an open neighborhood U of x_0 such