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# The effect of the domain topology on the number of positive solutions of an elliptic Kirchhoff problem

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ABSTRACT

Using minimax methods and Lusternik–Schnirelmann theory, we study multiple positive solutions for the Schrödinger–Kirchhoff equation

$$M\left(\int_{\Omega_\lambda}|\nabla u|^2dx+\int_{\Omega_\lambda}u^2dx\right)[-\Delta u+u]=f(u)$$

in $\Omega_\lambda=\lambda\Omega$. The set $\Omega\subset\mathbb{R}^3$ is a smooth bounded domain, $\lambda>0$ is a parameter, $M$ is a general continuous function and $f$ is a superlinear continuous function with subcritical growth. Our main result relates, for large values of $\lambda$, the number of solutions with the least number of closed and contractible in $\overline{\Omega}$ which cover $\overline{\Omega}$.



## 1. Introduction

In this paper we study multiple positive solutions for the following problem

$$\begin{cases}\mathcal{L}u=f(u), & \Omega_\lambda\\ u>0, & \Omega_\lambda\\ u=0, & \partial\Omega_\lambda\end{cases}\qquad (P_\lambda)$$

where $\Omega\subset\mathbb{R}^3$ is a smooth bounded domain, $\lambda>0$ is a parameter, $\Omega_\lambda:=\lambda\Omega$ is an expanding domain and $\mathcal{L}$ is the nonlocal operator given by

$$\mathcal{L}u=M\left(\int_{\Omega_\lambda}|\nabla u|^2dx+\int_{\Omega_\lambda}u^2dx\right)[-\Delta u+u].$$

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In 1883, Kirchhoff [1] established the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0 \tag{K}$$

where $L$ is the length of the string, $h$ is the area of cross-section, $E$ is the Young modulus of the material, $\rho$ is the mass density and $P_0$ is the initial tension. This model was proposed to modify the classical d'Alembert's wave equation, assuming a nonlinear dependence of the axial strain on the deformation of the gradient.

Owing to its importance in engineering, physics and material mechanics, a considerable effort has been devoted during the last years to the study the generalization of the stationary equation associated with problem $(K)$. With no hope of being thorough, we mention some papers regarding the study of this class of problems: [2–9] and reference therein. For an excellent didactic about this class of problems we cite [10] and for an overview of non-local problems we cite [11].

Problem $(P_\lambda)$ is a generalization of the stationary problem associated with problem $(K)$. Before stating our main result, we need the following hypotheses on the functions $M$ and $f$.

The continuous function $M:\mathbb{R}_+\to\mathbb{R}_+$ and the nonlinearity $f:\mathbb{R}\to\mathbb{R}$ satisfies the following conditions:

$(M_1)$ There is $m_0>0$ such that $M(t)\geq m_0,\ \forall t\geq 0$.
$(M_2)$ The function $t\mapsto M(t)$ is increasing.
$(M_3)$ The function $t\mapsto \frac{M(t)}{t}$ is decreasing.

A typical example of function verifying the assumptions $(M_1)$–$(M_3)$ is given by $M(t)=m_0+bt$, with $m_0>0$ and $b>0$. More generally, each function of the form $M(t)=m_0+bt+\sum_{i=1}^k b_it^{\gamma_i}$ with $b_i\geq 0$ and $\gamma_i\in(0,1)$ for all $i\in\{1,2,\ldots,k\}$ verifies the hypotheses $(M_1)$–$(M_3)$.

Now we give an example of a continuous but non-differentiable function that satisfies such hypotheses. Let $m_0,b_0,b_1$ and $t_0$ be positive constants such that $b_0\neq b_1$ and $t_0<\frac{m_0}{b_1-b_0}$ if $b_0<b_1$. We define the continuous function

$$M(t)=\begin{cases} m_0+b_0t, & \text{if } 0\leq t\leq t_0\\ m_0+(b_0-b_1)t_0+b_1t, & \text{if } t_0\leq t.\end{cases}$$

Since that $b_0\neq b_1$, we have that $M$ is non-differentiable in $t_0$. Using the same reasoning, we can build continuous functions that are not differentiable in a finite number of points.

We assume that the locally Lipschitz continuous function $f$ vanishes in $(-\infty,0)$ and verifies

$(f_1)$

$$\lim_{t\to0^+}\frac{f(t)}{t^3}=0.$$

$(f_2)$ There is $q\in(4,6)$ such that

$$\lim_{t\to\infty}\frac{f(t)}{t^{q-1}}=0.$$

$(f_3)$ There is $\theta\in(4,6)$ such that

$$0<\theta F(t)\leq f(t)t,\quad \forall t>0,$$

where $F(s)=\int_0^s f(t)dt$.