Nonlinear Analysis: Real World Applications 28 (2016) 290–299

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Extinction and blow-up phenomena in a non-linear gender structured population model

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ARTICLE INFO

Article history:
Received 18 February 2015
Received in revised form 30 September 2015
Accepted 10 October 2015
Available online 11 November 2015

Keywords:
Age structured population
Two sex models
Non-linear renewal equation
Blow up

ABSTRACT

In this article we consider a gender structured model in population dynamics. We assume that the fertility rate depends upon the weighted population of males instead of total population of males. The proportion of males in the population is determined by fixed environmental or social conditions. Here we prove an existence and uniqueness result for a non-trivial steady state. If the initial age distribution is uniformly below the non-trivial steady state then we show that the total population goes extinct in infinite time. On the other hand, if we take the initial age distribution to be uniformly above the steady state then the total population blows up exponentially with time.

1. Introduction

The modeling of gender structured population dynamics is usually focused on the mating process which leads to non-linear complex dynamics. These are more complicated to analyze than general age structured population dynamics. Work done by Darwin and Fisher contributed many mechanisms in the evolutionary perspective which determine the differences in the abundance of females and males in a population (see [1–3]). In [4], Iannelli et al. have modeled marriages and considered many gender structured population models.

The presence of having both female and male reproductive organs is called hermaphroditism. This is a mechanism for generating age-specific sex heterogeneity. In general, hermaphroditism can be classified into two types. They are simultaneous hermaphroditism and sequential hermaphroditism. In simultaneous hermaphroditism each individual produces both kinds of (female and male) gametes during its breading time. On the other hand, in the case of sequential hermaphroditism, an individual can change its gender only once during its life time. Sequential hermaphroditism is common in fishes (like teleost fish) and many gastropods. Broadly, sequential hermaphrodites are of two types, protandrous hermaphrodites: born as male

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http://dx.doi.org/10.1016/j.nonrwa.2015.10.005

and later change their sex to female, protogynous hermaphrodites: born as female and change their sex to male (see [1]).

In [5], Calsina et al. have modeled a sequential hermaphrodite population and stability of its steady states are analyzed. In addition, the same authors have studied the age at sex reversal from the evolutionary point of view by means of the function valued adaptive dynamics (see [6]). For a detailed discussion on stage structured population models and stage transitions see, for instance, the book by H.R. Thieme [7].

We assume that sex ratio is determined by fixed environmental or social factors. Let $\sigma(a)$ be the fixed proportion of males at age a in the population. In the context of sequential hermaphroditism one can consider $\sigma(a)$ as the probability of being male at age a (see [8]). Therefore, $\sigma(0)$ represents the probability of being born as a male. One can interpret the case $\sigma(0) > 0$ in two ways. First, some individuals change their sex instantaneously when they are born. Second, there are two kinds of males in the population namely, primary males and secondary males. Primary males are those individuals which are born as males and do not change sex. In contrary, secondary males are born as females and have changed their sex to become males (see [5]).

Let $u(t,a)$ be the density with respect to age a at time t of a closed population. Assume that $a_\dagger$ is the maximum age of survival of the species. Moreover, we assume that the effects of population density that are felt by individuals are dependent on that individual's class, *i.e.,* age, such that the vital rates depend on the weighted population with weights $\psi_1(a),\ \phi_2(a)$. Let d be the mortality rate of the joint sex population. We assume that d depends on age of each individual and a weighted total population with weight $\psi_1(a)$.

We assume that the birth process is modeled taking into account the mechanism of encounters (or interactions) between males and females. To this end, we introduce a function $B_1(a,S)$ that gives functional response to the search of male partner, in other words, it represents the number of males that a female of age a encounters per unit time. On the other hand, let $B_2(a)$ be the intrinsic age-specific fertility function, which denotes the average number of offspring produced per an encounter by a female of age a. Therefore, the total number of newborns that are produced at time t is

$$\int_0^{a_\dagger} B_2(a)B_1\big(a,S_2(t)\big)u(t,a)\big(1-\sigma(a)\big)da,$$

where $S_2(t)$ is the weighted population of males with weight $\phi_2(a)$ at time t. Define,

$$B(a,S_2(t)) := B_2(a)B_1(a,S_2(t))u(t,a)(1-\sigma(a)), \qquad \psi_2(a) := \sigma(a)\phi_2(a).$$

With this notation and assumptions the dynamics of the population are governed by the following McKendrick–von Foerster system,

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,a) + \dfrac{\partial}{\partial a}u(t,a) + d\big(a,S_1(t)\big)u(t,a) = 0, \quad t>0,\ 0<a<a_\dagger, \\ u(t,0) = \displaystyle\int_0^{a_\dagger} B\big(a,S_2(t)\big)u(t,a)da, \quad t>0, \\ u(0,a) = u_0(a) \ge 0, \qquad u_0(.) \in L^1([0,a_\dagger]) \cap L^\infty([0,a_\dagger]), \end{cases} \tag{1}$$

where

$$S_i(t) = \int_0^{a_\dagger} \psi_i(a)u(t,a)da \quad \text{for } i=1,2. \tag{2}$$

The existence and uniqueness of a weak solution to (1)–(2) is discussed in [9–11]. The stability analysis of steady states of this system of equations has attracted interest of many people (see [12,9,11] and the references given there). In [13–15], Perthame et al. introduced the concept of General Relative Entropy to study the stability of linear version of the above equation. In [10], Perthame et al. have converted the non-linear renewal equation into a system of ordinary differential equations in some cases and the stability of the non-trivial steady state has been discussed. Linear stability around the non-trivial steady state is considered